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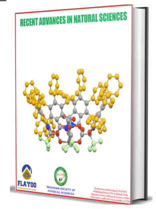

Hybrid block formulae whose eigenvalues of Jacobian matrices are close to the imaginary axis of the complex plane

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ABSTRACT

Methods for integrating stiff initial value problems are required to be A-stable. Of great interest are A-stable methods whose Jacobian matrices have their eigenvalues close to the imaginary axis of the complex plane. This class of A-stable methods are very rare. This paper is on the development of a new family of A-stable hybrid block method whose Jacobian matrices possess eigenvalues on the imaginary of the complex plane via interpolation and collocation techniques. The family of methods developed herein are A-stable for order $p \leq 18$. Numerical solutions generated by the new method are compared with existing methods in the literature. The numerical results show that the new class of methods are more efficient and accurate.

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1. INTRODUCTION

Physical phenomena are often modelled in science and engineering using differential equations, some of these physical phenomena includes: the determination of motion of planetary bodies, change in population size of species over a period of time and the rate of decay of radioactive elements. These models often give rise to differential equations, most of which are insoluble using analytic techniques. Hence the need for numerical methods. Stiff equations are a subset of differential equations that require special numerical methods due to their inherent instability unless very small step sizes are used.

Analytical solutions for stiff problems are often challenging or impossible to obtain, making numerical methods crucial for solving them. Stiff problems occurs when one or more components of solution of a differential equations decay much more rapidly than others. Stiffness also occur when stability requirement rather than accuracy constrains the step-length for numerical integration. Stiffness is when the eigen values of the jacobian of the system of ordinary differential equations (ODEs) differs greatly in magnitude Ref. [1]. Jacobian matrices need to have eigenvalues close to the imaginary axis because by ensuring that the jacocian eigenvalues are close to the imaginary axis, these methods maintain stability while accurately capturing the behavior of the system Ref. [2]. and this type of methods are rare and usually implicit. In general, eigenvalues of a matrix are not typically located on the imaginary axis of the complex plane. They can be real or complex, but not necessarily on the imaginary axis. A-stable methods are designed to be stable for certain types of equations, and having eigenvalues on the imaginary axis may not be consistent with these properties. A good numerical method is that which is accurate,fast in computation and produces minimum/low error.

This study is on the development of new class of hybrid-block

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formulae that are stable and suitable for solving stiff system of Initial Value Problems (IVP) in Ordinary Differential equations (ODEs) of the form:

$$y' = f(t, y),\ y(a) = \eta,$$
$$f : \Re \times \Re^m \to \Re^m,\ y : \Re \to \Re^m\ t \in [a, b]. \quad (1)$$

The development of methods for solving equation (1) is still receiving research attention as evident in literature. Ref. [3] presented a class of extended backward differentiation formular suitable for the approximation of stiff systems of 1st order ordinary differential equations. The family of method presented herein was L-stable up to order 4. It has the advantages of ease of changing order and size, highly stable and relatively low computational effort per step. Also Ref. [4] focussed on second derivative Multistep methods for stiff systems. Since stiff systems are characterized by rapid variations in their solutions and specialised numerical methods are required to handle them effectively, the method developed in Ref. [4] aimed at enhancing accuracy and stability of numerical solutions for stiff IVPs. The introduction of a family of methods in Ref. [5] designed for solving stiff IVPs in ODEs are inherently parallel and can be distributed across parallel processors. This new restructured methods are zero stable apriori. They exhibit A-stablilty for block sizes up to four and are well suited for integration of stiff IVPs on parallel computers. Error analysis and stability bounds were discussed in Ref. [6]. Author investigated the accuracy and robustness of numerical integration schemes, emphasising the need for stable methods to ensure reliable results. Its findings has continue to guide researchers in designing efficient and accurate stiff ODEs solvers. Ref. [7] delved into various classes of step by step methods suitable for automatic numerical integration of general problems. Also, it covers the derivation of methods, the theory of error and convergence and practical implementation on a computer. In Ref. [8] a diagonally implicit Runge Kutta methods that approximate solutions to ODEs by implicitly solving a system of equation at each stage was developed. This family of methods developed herein are characterised by their stage order, accuracy, L-stability and embedded dense-output capabilities. Also Ref. [9] derived a block of diagonally implicit backward differentiation method with two off step points. This proposed method computes two approximate solution values concurrently for every iteration and it was found to be zero stable, A-stable and suitable for solving first order stiff IVPs. While Ref. [10] proposed a novel higher-order hybrid super class backward differentiation formula for simulating stiff IVPs of ODEs. The new scheme approximates the values of two points and two off step points per integration step. and also incorporate a stability control parameter by varying the free parameter p within interval $(-1, 1)$, different zero stable and A-stable schemes can be obtained.

The core characteristic of stiffness lies in the fact that the solution components of IVPs to be computed exhibits different variations, yet it is accompanied by rapidly damped perturbations.

Formulae that incorporates function evaluations at intra step points are known to have higher order A-stable members compared to classical Linear Multistep Methods. Methods that allow intra-step computations are known as Hybrid Refs. [11–13]. The term 'hybrid' was coined in Ref. [13] since these methods while preserving certain Linear Multistep Methods (LMM) characteristics, also share with Runge-Kutta Methods (RKM)the ability to utilize data from intra-step points other than the primary step points. Linear Multistep Methods can be composed to form what is known in literature as block methods. Block methods have the characteristics of generating approximate solution at different points simultaneously.

A general r-block , k-point block method is a matrix finite difference equation of the form

$$A_0 Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + h \sum_{i=0}^{k} B_i F_{m-i}, \quad (2)$$

where $A_0 = I_k$, $A_i's$ and $B_i's$ are coefficient in $k \times k$ matrix form and $m = 0, 1, \cdots$ represents the member of blocks, $n = mk$ is the first number of the m^{th} block and k is the proposed block size.

$$Y_m = (y_{n+1}, y_{n+2}, \cdots, y_{n+k})^T, \quad (3)$$
$$F_m = (f_{n+1}, f_{n+2}, \cdots, f_{n+k})^T, \quad (4)$$
$$Y_{m-i} = (y_{n-ik+1}, y_{n-ik+2}, \cdots, y_{n-ik+k})^T, i = 1, 2, \cdots, k, \quad (5)$$
$$F_{m-i} = (f_{n-ik+1}, f_{n-ik+2}, \cdots, f_{n-ik+k})^T, i = 1, 2, \cdots, k. \quad (6)$$

For $k = 1$, equation (2) reduces to the classical r-step LMM. For $B_0 = 0$, the block method equation (2) is explicit, otherwise it is implicit. A strictly lower triangular matrix B_0, equation (2) is also explicit. When $r = 1$ in equation (2) results into one block k-point method.

A one block method is defined by the recurrence equation given as:

$$Y_{n+1} = AY_n + hBF(Y_n) + hDF(Y_{n+1}), \quad (7)$$

where A, B and D are $k \times k$ matrices. One-Block method of equation (7) is suitable for parallel implementation, if the matrix D is diagonal. Block methods of different form developed in Ref. [1, 14, 15] are suitable for solving equation (1).

In Ref. [16], a new two-step hybrid block method (IMBLOCK) for the numerical integration of ordinary differential initial value systems was presented. The method is obtained when the two intermediate points and the approximation of the real solution with a sufficient polynomial and the collocation conditions are determined. The proposed method has an algebraic order of tenth convergence and is A-stable.

Ref. [17] developed a class of $A(\alpha)$ stable linear multistep methods for solving stiff problems. The methods are A-stable for $k \leq 3$ and rigidly stable for $4 \leq k \leq 6$. They solved the resulting nonlinear system using the Newton-Raphson formula. Also Ref. [18] focused on the derivation of the Sixth Partial Hybrid Block Method (OSHBM) for the general solution of first-order initial value problems of ordinary differential equations. The new proposed method was derived using the Chebyshev polynomial collocation and interpolation approach, which is an approximate solution to obtain a continuous linear multistep method at some selected points, which was analyzed in some off line hybrid linear multistep methods. The main features of the proposed method were investigated and it was found that the method is zero-stable, consistent and convergent but it has a setback of low order. The

OSHBM has an order of $p \leq 4$. Then Ref. [19] developed 3-step hybrid Adams-type method (HATM) to solve a first-order ordinary differential equation (ODE).

Diagrams derived at both on-line and off-line points using multi-step collocation methods, and some points were also analyzed using the Block Adams-type method and Adams Moulton method, respectively. The highest ranked method was selected as the corrector. The merger was valid and effective. Numerical experiments were performed and it was found that Adams-type hybrid methods perform better than the traditional Adams-Moulton method but it has a setback of low order. Ref. [16] presented a new two-step hybrid block method (IMBLOCK) for the numerical integration of ordinary differential initial value systems. The method is obtained when the two intermediate points and the approximation of the real solution with a sufficient polynomial and the collocation conditions are determined. The proposed method has an algebraic order of tenth convergence and is A-stable. Also Ref. [20] derive a block of diagonally implicit backward differentiation method for solving first-order stiff IVPs in which two approximate solution values are computed concurrently for each iteration at two off step points. The new method was found to be zero stable, A-stable and even perform better in terms of accuracy when compared to other methods that was reviewed. Ref. [21] proposed a novel block of higher-order hybrid super class backward differentiation formula (HSBBDF) for simulating stiff initial value problems (IVPs) of ordinary differential equations (ODEs).This family of method belongs to the super class of BDFs and includes a stability control parameter. By varying the value of the free parameter within a certain interval, different zero-stable and A-stable schemes can be obtained. The specific choice of order results in a zero- stable and A-stable method capable of solving stiff IVPs of ODEs.

This research aimed at developing a new Family of hybrid block method Known as Modified Continous Hybrid-Type Formulae(MCHTF) by the method of continous interpolation (a point where the solution of a function is evaluated) and collocation (a point where the derivative of a function is evaluated) with very high order and also possesses zero and A- stable properties that are required to solve a stiff IVP of ODEs. The uniqueness of the method developed herein is that it's stability are on the imaginary axis of the complex plain hence they are perfectly Stable and can solve most stiff problems. High order methods ensure that the pertubations of the solution do not diverge away over time. They are known to be efficient, accurate and can solve wide range of problems and are less sensitive to errors

2. METHOD FORMULATION

Consider a continuous hybrid method of the form

$$\sum_{j=0}^{2k} \beta_{i,j} y_{n+v_j} - h\beta_{2k+1} f_{n+v_{i-1}} = hf_{n+v_i},$$
$$v_i = \frac{i}{2},\ i = 1, 2, \cdots, 2k, \tag{8}$$

on an intra-step points $t_0, t_{\frac{1}{2}}, t_1, t_{\frac{2k-1}{2}}, t_k$, where y_{n+v_i} is the discrete approximation of the analytic solution $y(t_n + v_i)$ at t_{n+v_i}, $f_{n+v_i} = f(t_n + v_i, y_n + v_i)$, h is the given stepsize and the $\beta_{i,j}s$ are coefficients that are determined through the intra-step point using interpolation and collocation approach. The method equation (8) can be compose to give block method. Thus, the method in equation (8) is obtained by approximating a basis polynomial of the form:

$$y(t) = \sum_{j=0}^{2k+1} \beta_j \left(\frac{t - t_n}{h}\right)^j, \tag{9}$$

$$y'(t) = \frac{1}{h}\sum_{j=0}^{2k+1} j\beta_j \left(\frac{t - t_n}{h}\right)^{j-1}. \tag{10}$$

Since the normalization of the coefficients in equation (8) occur in the first derivative part, the basis polynomial in equation (10) shall be used to derived the continuous method equation (8) through the means of interpolating $y(t)$ at point t_{n+v_j} and collocating $y'(t)$ at t_{n+v_i}, $i, j = 0(1)k$. That is

$$\begin{aligned} y(t_{n+v_j}) &= y_{n+v_j}; \quad j = 0, 1, 2, 3, 2k \\ y'(t_{n+v_i}) &= y'_{n+v_i}; \quad i =, 1, 2, 3, \cdots, 2k, \\ y'(t_{n+v_{i-1}}) &= y'_{n+v_{i-1}}; \quad i =, 1, 2, 3, 2k. \end{aligned} \tag{11}$$

This leads to a system of $2k + 2$ linear equations given in a compact form as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & v_1 & v_1^2 & v_1^3 & \cdots & v_1^{2k} & v_1^{2k+1} \\ 1 & v_2 & v_2^2 & v_2^3 & \cdots & v_2^{2k} & v_2^{2k+1} \\ 1 & v_3 & v_3^2 & v_3^3 & \cdots & v_3^{2k} & v_3^{2k+1} \\ \vdots & & & & & \vdots & \\ 1 & v_{2k} & v_{2k}^2 & v_{2k}^3 & \cdots & v_{2k}^{2k} & v_{2k}^{2k+1} \\ 0 & 1 & 2v_{i-1} & 3v_{i-1}^2 & \cdots & 2kv_{i-1}^{2k-1} & (2k+1)v_1^{2k} \end{pmatrix} \times \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{2k} \\ \beta_{2k+1} \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+v_1} \\ y_{n+v_2} \\ y_{n+v_3} \\ \vdots \\ y_{n+v_{2k}} \\ hf_{n+v_{i-1}} \end{pmatrix}. \tag{12}$$

The system of equation (12) is used to determine the coefficients β_i, j, for continuous methods:

$$\sum_{j=0}^{2k} \beta_{i,j} y_{n+v_j} - h\beta_{2k+1} y\prime(t_{n+v_{i-1}}) = hy\prime(t_{n+v_i}), \quad i = 1, 2, \cdots, 2k, \tag{13}$$

for $i = 1, 2, \cdots, 2k$ in equation (7) gives the hybrid-block formula. The new continous schemes equation (8) has the characteristics of Runge-Kutta method of being self-starting. The method equation (8) is composed into a one block method of the form:

$$A_1 Y_{n+1} = A_0 Y_n + h\left(B_1 F_{n+1} + B_0 F_n\right), \tag{14}$$

where the $k \times k$ matrices coefficient are defined as:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \beta_{2,2k+1} & 1 & 0 & \ddots & 0 & 0 \\ 0 & \beta_{3,2k+1} & 1 & \ddots & \vdots & 0 \\ 0 & 0 & \beta_{3,2k+1} & \ddots & 0 & 0 \\ & & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ddots & \beta_{2k,2k+1} & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \beta_{1,2k+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{15}$$

$$A_1 = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,2k} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,2k} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \beta_{2k-1,1} & \beta_{2k-1,2} & \cdots & \beta_{2k-1,2k} \\ \beta_{2k,1} & \beta_{2k,2} & \cdots & \beta_{2k,2k} \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \beta_{1,0} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{2,0} \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \\ & \vdots & & \ddots & 0 & \beta_{2k-1,0} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{2k,0} \end{pmatrix}, \tag{16}$$

and the block solution output and its derivative are, respectively

$$\begin{aligned} Y_{n+1} &= \left[y_{n+v_1}, y_{n+v_2}, \cdots, y_{n+v_{2k}}\right]^T, \\ F_{n+1} &= \left[f_{n+v-1}, f_{n+v_2}, \cdots, f_{n+v_{2k}}\right]^T. \end{aligned} \tag{17}$$

2.1. ONE BLOCK SIX-POINTS METHOD

The intra-step points for $k = 3$ in equation (14) are $t_0, t_{\frac{1}{2}}, t_1, t_{\frac{3}{2}}, t_2, t_{\frac{5}{2}}, t_3$ and the resultant continuous schemes is given as

$$\begin{aligned} &-\frac{23y_n}{20} - \frac{17}{30}y_{n+\frac{1}{2}} + \frac{5y_{n+1}}{2} - \frac{10}{9}y_{n+\frac{3}{2}} + \frac{5y_{n+2}}{12} - \frac{1}{10}y_{n+\frac{5}{2}} + \frac{y_{n+3}}{90} \\ &\qquad = \frac{h}{6}\left(f_n + 6f_{n+\frac{1}{2}}\right), \\ &-\frac{y_n}{15} - \frac{137}{75}y_{n+\frac{1}{2}} + \frac{5y_{n+1}}{6} + \frac{4}{3}y_{n+\frac{3}{2}} - \frac{y_{n+2}}{3} + \frac{1}{15}y_{n+\frac{5}{2}} - \frac{y_{n+3}}{150} \\ &\qquad = \frac{h}{5}\left(5f_{n+1} + 2f_{n+\frac{1}{2}}\right), \\ &\frac{y_n}{45} - \frac{2}{5}y_{n+\frac{1}{2}} - \frac{19y_{n+1}}{6} + \frac{8}{3}y_{n+\frac{3}{2}} + y_{n+2} - \frac{2}{15}y_{n+\frac{5}{2}} + \frac{y_{n+3}}{90} \\ &\qquad = \frac{h}{3}\left(3f_{n+1} + 4f_{n+\frac{3}{2}}\right), \\ &\frac{19y_{n+2}}{8} + \frac{1}{10}y_{n+\frac{1}{2}} - 2y_{n+\frac{3}{2}} + \frac{3}{10}y_{n+\frac{5}{2}} - \frac{3y_{n+1}}{4} - \frac{y_{n+3}}{60} - \frac{y_n}{120} \\ &\qquad = \frac{h}{4}\left(4f_{n+\frac{3}{2}} + 3f_{n+2}\right), \end{aligned} \tag{18}$$

$$\begin{aligned} &+\frac{y_n}{150} - \frac{1}{15}y_{n+\frac{1}{2}} + \frac{y_{n+1}}{3} - \frac{4}{3}y_{n+\frac{3}{2}} - \frac{5y_{n+2}}{6} + \frac{137}{75}y_{n+\frac{5}{2}} + \frac{y_{n+3}}{15} \\ &\qquad = \frac{h}{5}\left(5f_{n+2} + 2f_{n+\frac{5}{2}}\right), \\ &-\frac{y_n}{90} + \frac{1}{10}y_{n+\frac{1}{2}} - \frac{5y_{n+1}}{12} + \frac{10}{9}y_{n+\frac{3}{2}} - \frac{5y_{n+2}}{2} + \frac{17}{30}y_{n+\frac{5}{2}} + \frac{23y_{n+3}}{20} \\ &\qquad = \frac{h}{6}\left(6f_{n+\frac{5}{2}} + f_{n+3}\right), \end{aligned} \tag{19}$$

which form the modified continous hybrid-type formula in equation (14) with coefficients given as:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{23}{20} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{45} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{120} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{150} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{90} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\frac{17}{30} & \frac{5}{2} & -\frac{10}{9} & \frac{5}{12} & -\frac{1}{10} & \frac{1}{90} \\ -\frac{137}{75} & \frac{5}{6} & \frac{4}{3} & -\frac{1}{3} & \frac{1}{15} & -\frac{1}{150} \\ -\frac{2}{5} & -\frac{19}{6} & \frac{8}{3} & 1 & -\frac{2}{15} & \frac{1}{90} \\ \frac{1}{10} & -\frac{3}{4} & -2 & \frac{19}{8} & \frac{3}{10} & -\frac{1}{60} \\ -\frac{1}{15} & \frac{1}{3} & -\frac{4}{3} & -\frac{5}{6} & \frac{137}{75} & \frac{1}{15} \\ \frac{1}{10} & -\frac{5}{12} & \frac{10}{9} & -\frac{5}{2} & \frac{17}{30} & \frac{23}{20} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{5} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{6} \end{pmatrix}. \tag{20}$$

2.1.1. Coefficients of one block eight-points method

The coefficients of one block eight-points method are given below:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1041}{1120} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{84} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{336} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{840} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{588} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{224} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\frac{83}{70} & \frac{7}{2} & -\frac{7}{3} & \frac{35}{24} & -\frac{7}{10} & \frac{7}{30} & -\frac{1}{21} & \frac{1}{224} \\ -\frac{363}{245} & \frac{1}{10} & 2 & -\frac{5}{6} & \frac{1}{3} & -\frac{1}{10} & \frac{2}{105} & -\frac{1}{588} \\ -\frac{2}{7} & -\frac{39}{10} & \frac{11}{5} & \frac{5}{2} & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{35} & \frac{1}{420} \\ \frac{1}{21} & -\frac{1}{2} & -\frac{29}{10} & \frac{5}{2} & 1 & -\frac{1}{6} & \frac{1}{42} & -\frac{1}{560} \\ -\frac{2}{105} & \frac{2}{15} & -\frac{4}{5} & -2 & \frac{58}{25} & \frac{2}{5} & -\frac{4}{105} & \frac{1}{420} \\ \frac{1}{70} & -\frac{1}{12} & \frac{1}{3} & -\frac{5}{4} & -\frac{11}{10} & \frac{39}{20} & \frac{1}{7} & -\frac{1}{168} \\ -\frac{2}{105} & \frac{1}{10} & -\frac{1}{3} & \frac{5}{6} & -2 & -\frac{1}{10} & \frac{363}{245} & \frac{1}{28} \\ \frac{1}{21} & -\frac{7}{30} & \frac{7}{10} & -\frac{35}{24} & \frac{7}{3} & -\frac{7}{2} & \frac{83}{70} & \frac{1041}{1120} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0&0&0&0&0&0&0&\frac{1}{8}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ \frac{2}{7}&1&0&0&0&0&0&0\\ 0&1&2&0&0&0&0&0\\ 0&0&1&\frac{5}{4}&0&0&0&0\\ 0&0&0&1&\frac{4}{5}&0&0&0\\ 0&0&0&0&1&\frac{1}{2}&0&0\\ 0&0&0&0&0&1&\frac{2}{7}&0\\ 0&0&0&0&0&0&1&\frac{1}{8} \end{pmatrix}. \tag{21}$$

2.1.2. Coefficients of one block ten-points method

$$A_1 = \begin{pmatrix} -\frac{2089}{1260}&\frac{9}{2}&-4&\frac{7}{2}&-\frac{63}{25}&\frac{7}{5}&-\frac{4}{7}&\frac{9}{56}&-\frac{1}{36}&\frac{1}{450}\\ -\frac{7129}{5670}&\frac{61}{140}&\frac{8}{3}&-\frac{14}{9}&\frac{14}{15}&-\frac{7}{15}&\frac{8}{45}&-\frac{1}{21}&\frac{1}{126}&-\frac{1}{1620}\\ -\frac{2}{9}&-\frac{621}{140}&\frac{404}{315}&\frac{14}{3}&-\frac{28}{15}&\frac{7}{9}&-\frac{4}{15}&\frac{1}{15}&\frac{2}{189}&\frac{1}{1260}\\ \frac{1}{36}&-\frac{3}{8}&-\frac{739}{210}&\frac{133}{60}&\frac{21}{10}&-\frac{7}{12}&\frac{1}{6}&-\frac{3}{80}&\frac{1}{180}&-\frac{1}{2520}\\ -\frac{1}{126}&\frac{1}{14}&-\frac{4}{7}&-\frac{41}{15}&\frac{12}{5}&1&-\frac{4}{21}&\frac{1}{28}&-\frac{1}{210}&\frac{1}{3150}\\ \frac{1}{252}&-\frac{5}{168}&\frac{10}{63}&-\frac{5}{6}&-2&\frac{41}{18}&\frac{10}{21}&-\frac{5}{84}&\frac{5}{756}&-\frac{1}{2520}\\ -\frac{1}{315}&\frac{3}{140}&-\frac{2}{21}&\frac{1}{3}&-\frac{6}{5}&-\frac{19}{15}&\frac{1478}{735}&\frac{3}{14}&-\frac{1}{63}&\frac{1}{1260}\\ \frac{1}{252}&-\frac{1}{40}&\frac{1}{10}&-\frac{7}{24}&\frac{7}{10}&-\frac{7}{4}&-\frac{101}{210}&\frac{1863}{1120}&\frac{1}{12}&-\frac{1}{360}\\ -\frac{1}{126}&\frac{1}{21}&-\frac{8}{45}&\frac{7}{15}&-\frac{14}{15}&\frac{14}{9}&-\frac{8}{3}&\frac{61}{140}&\frac{7129}{5670}&\frac{1}{45}\\ \frac{1}{36}&-\frac{9}{56}&\frac{4}{7}&-\frac{7}{5}&\frac{63}{25}&-\frac{7}{2}&4&-\frac{9}{2}&\frac{2089}{1260}&\frac{9901}{12600} \end{pmatrix}, \tag{22}$$

$$A_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&-\frac{9901}{12600}\\ 0&0&0&0&0&0&0&0&0&-\frac{1}{45}\\ 0&0&0&0&0&0&0&0&0&\frac{1}{135}\\ 0&0&0&0&0&0&0&0&0&-\frac{1}{720}\\ 0&0&0&0&0&0&0&0&0&\frac{1}{2100}\\ 0&0&0&0&0&0&0&0&0&-\frac{1}{3780}\\ 0&0&0&0&0&0&0&0&0&\frac{1}{4410}\\ 0&0&0&0&0&0&0&0&0&-\frac{1}{3360}\\ 0&0&0&0&0&0&0&0&0&-\frac{1}{126}\\ 0&0&0&0&0&0&0&0&0&\frac{1}{36} \end{pmatrix}, \tag{23}$$

$$B_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ \frac{2}{9}&1&0&0&0&0&0&0&0&0\\ 0&1&\frac{8}{3}&0&0&0&0&0&0&0\\ 0&0&1&\frac{7}{4}&0&0&0&0&0&0\\ 0&0&0&1&\frac{6}{5}&0&0&0&0&0\\ 0&0&0&0&1&\frac{5}{6}&0&0&0&0\\ 0&0&0&0&0&1&\frac{4}{7}&0&0&0\\ 0&0&0&0&0&0&1&\frac{3}{8}&0&0\\ 0&0&0&0&0&0&0&1&\frac{2}{9}&0\\ 0&0&0&0&0&0&0&0&1&\frac{1}{10} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&\frac{1}{10}\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{24}$$

3. STABILITY ANALYSIS

The constituent hybrid linear multistep methods that form the equation (8) are obtained from:

$$\sum_{j=0}^{2k} \beta_{i,j} y(t+v_j h) - h\left(\beta_{i,2k+1} y'(t+v_{i-1}h) - y'(t+v_i h)\right) = C_{2k+2} h^{2k+2} y^{(2k+2)}(t) + O\left(h^{2k+3}\right). \tag{25}$$

Linear difference operator $L[y(t);h]$ with equation (8) given by

$$L[y(t);h] = \sum_{j=0}^{2k} \beta_{i,j} y(t+v_j h) - h\left(\beta_{i,2k+1} y'(t+v_{i-1}h) - y'(t+v_i h)\right), \tag{26}$$

where $y(t)$ is assumed to be sufficiently differentiable function on $[a,b]$. The equation (26) provides the local truncation error (lte) of the hybrid block scheme in equation (14). By expanding via Taylor series the term $y'(t+v_i h)$ and $y(t+jh)$ about t and collecting terms by the power of h, equation (26) becomes:

$$L(y(t);h) = \bar{C}_0 y(t) + \bar{C}_1 h y'(t) + \cdots + \bar{C}_p h^p y^p(t) + \cdots, \tag{27}$$

where

$$\bar{C}_0 = \sum_{j=0}^{2k} \beta_{i,j}, \bar{C}_1 = \sum_{j=0}^{2k} j\beta_{i,j} - \beta_{i,2k+1} - 1,$$

$$\bar{C}_2 = \sum_{j=0}^{2k} \frac{j^2}{(2)!}\beta_{i,j} - (v_{i-1})\beta_{i,2k+1} - v_i,$$

$$\vdots$$

$$\bar{C}_p = \frac{(v_i)^p}{p!} - \sum_{j=0}^{2k} \frac{j^p}{(p)!}\beta_{i,j} - \frac{j^{p-1}}{(p-1)!}\left((v_{i-1})\beta_{i,2k+1} + v_i\right),$$
$$p \geq 1, \quad k \geq 1, \tag{28}$$

with $\bar{C}_p = \left[C_{1,p}, C_{2,p}, \cdots, C_{2k,p}\right]^T$.

The hybrid scheme of equation (8) is of order p, if

$$\bar{C}_j = 0, \quad j = 0(1)p, \quad \bar{C}_{p+1} \neq 0, \tag{29}$$

where $\bar{C}_{p+1} \neq 0$ is the error constant (EC) and its principal local truncation error (LTE) is

$$LTE = \bar{C}_{p+1} h^{p+1} Y^{p+1} + O(h^{p+2}). \tag{30}$$

The error constants for $k = 3, 4, 5$ are given below:

$$C_6 = \begin{pmatrix} \frac{1}{43008}\\ -\frac{1}{107520}\\ \frac{1}{107520}\\ -\frac{1}{143360}\\ \frac{1}{107520}\\ -\frac{1}{43008} \end{pmatrix}, \tag{31}$$

$$C_8 = \begin{pmatrix} \frac{1}{368640} \\ -\frac{1}{1290240} \\ \frac{1}{1290240} \\ -\frac{1}{2580480} \\ \frac{1}{3225600} \\ -\frac{1}{2580480} \\ \frac{1}{1290240} \\ -\frac{1}{368640} \end{pmatrix}, \tag{32}$$

$$C_{10} = \begin{pmatrix} \frac{1}{2703360} \\ -\frac{1}{12165120} \\ \frac{1}{12165120} \\ -\frac{1}{32440320} \\ \frac{1}{56770560} \\ -\frac{1}{68124672} \\ \frac{1}{56770560} \\ -\frac{1}{32440320} \\ \frac{1}{12165120} \\ -\frac{1}{2703360} \end{pmatrix}. \tag{33}$$

Lemma 1. *Suppose the sequence $\{d_{i+1}\}$ satisfies the conditions*

$$d_{i+1} \leq (1 + \alpha h_{i+1})\, d_i + \sigma_{i+1} h_{i+1}; \quad i = 0, 1, \tag{34}$$

with the sequences $\{d_{i+1}\}$, $\{\sigma_{i+1}\}$, $\{h_{i+1}\}$ and α are non-negative integer, then

$$d_{i+1} \leq \left(d_0 + \sum_{j=0}^{i} \sigma_j h_j \right) \exp\left(\alpha \sum_{r=0}^{i} h_r \right). \tag{35}$$

Theorem 3.1. *Suppose the role of round-off error is negligible and the hybrid block scheme in equation (14) satisfies the Lipschitz condition*

$$\| F(t, y) - F(t, \bar{y}) \|_\infty \leq L \| y - \bar{y} \|_\infty, \tag{36}$$

for all $t \in [t_0, T]$ and $y, \bar{y}, \in \mathbb{C}$. The hybrid block methods in equation (14) is convergent of order $p = 2k + 1$, if is $\| A^{-1} A_0 \|_\infty = 1$ and consistent holds.

Proof.
Due to one block method in equation (14), the local truncation error is given as:

$$\tau_{n+1}(h) = A\bar{Y}_{n+1} - hB\bar{F}_{n+1} + A_0\bar{Y}_n - hB_0\bar{F}_n, \tag{37}$$

where

$$\begin{aligned} Y_{n+1} &= (y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, \cdots, y_{n+k})^T, \\ F_{n+1} &= (f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, \cdots, f_{n+k})^T, \end{aligned} \tag{38}$$

is the block of solution and function values. By subtracting equation (14) from (37) gives the global truncation error

$$\begin{aligned} \varepsilon_{n+1} = \bar{Y}_{n+1} - Y_{n+1} &= A^{-1}\tau_{n+1}(h) - A^{-1}A_0(\bar{Y}_n - Y_n) \\ &+ hA^{-1}B(\bar{F}_{n+1} - F_{n+1}) + hA^{-1}B_0(\bar{F}_n - F_n). \end{aligned} \tag{39}$$

For proper notation, let $\| (A^{-1}B \|_\infty = \varphi$, $\| A^{-1}\bar{B}_0 \|_\infty = \vartheta$, $e_0 = 0$, $d_{n+1} = \max_{0 \leq j \leq n} \| \varepsilon_{n+1} \|_\infty$, $n = 0(1)W_t$.

Thus, since the one-block method in equation (14) is pre-consistent that is $\left(\| A^{-1}A_0 \|_\infty = 1\right)$, then, the schemes in equation (14) is consistent for order $p = 2k + 1$. Following equation (36), this leads to

$$\begin{aligned} &\| \varepsilon_{n+1} \|_\infty = \| \varepsilon_n \|_\infty + Lh(\varphi \| \varepsilon_{n+1} \|_\infty \\ &+ \vartheta \| \varepsilon_n \|) + \| A^{-1} \|_\infty \| \tau_{n+1}(h) \|_\infty \\ &\leq d_i + Lh(\varphi d_{i+1} + \vartheta) d_i) + u \| A^{-1} \|_\infty h^{2k+2}, \end{aligned} \tag{40}$$

where $u > 0$ does not dependent on h and $n = 0(1)W_t$. Assume there exist a positive value h_0, and $L(\varphi h_0) < 1$ such that

$$\begin{aligned} &d_{n+1} \leq \left(\frac{1 - L(\varphi(h_0 - h) - h\vartheta)}{1 - L(\varphi h_0)} \right) d_n + \frac{Jh^{2k+2}}{1 - L(\varphi h_0)}, \\ &0 < h \leq h_0. \end{aligned} \tag{41}$$

Then, from lemma equation (1), we have

$$d_{n+1} \leq \frac{JQ}{k(1 - L(\varphi h_0))} \exp\left[\frac{L(\varphi + \vartheta)}{k(1 - L\varphi h_0)} \right] h^{2k+2}, \tag{42}$$

where $J = u \| (A^{-1} \|_\infty$ and $Q = W_t \widehat{h} = W_t Ns \cdot h$. Finally,

$$\max_{1 \leq n \leq W_t} \| \varepsilon_{n+1} \|_\infty \equiv O(h^{2k+2}). \tag{43}$$

□

3.1. ZERO STABILITY

In this section, the zero stability property of equation (14) is carried out. The first characteristics polynomial associated with the method equation (14) is given as:

$$\bar{\rho}(r) = A_1 r - A_0. \tag{44}$$

The method equation (14) is said to be zero stable if the roots of the first characteristic equation has all roots inside the unit circle and only one root on the boundary of the unit circle of the complex plain. Ref. [22].

To confirm the claim, we illustrate with the following. Given that determinant of $(A_1 r - A_0) = 0$ in equation (14). For $k = 3$ we have that

$$\left| \begin{pmatrix} -\frac{17}{30} & \frac{5}{2} & -\frac{10}{9} & \frac{5}{12} & -\frac{1}{10} & \frac{1}{90} \\ -\frac{137}{75} & \frac{5}{6} & \frac{4}{3} & -\frac{1}{3} & \frac{1}{15} & -\frac{1}{150} \\ -\frac{2}{5} & -\frac{19}{6} & \frac{8}{3} & 1 & -\frac{2}{15} & \frac{1}{90} \\ \frac{1}{10} & -\frac{3}{4} & -2 & \frac{19}{8} & \frac{3}{10} & -\frac{1}{60} \\ -\frac{1}{15} & \frac{1}{3} & -\frac{4}{3} & -\frac{5}{6} & \frac{137}{75} & \frac{1}{15} \\ \frac{1}{10} & -\frac{5}{12} & \frac{10}{9} & -\frac{5}{2} & \frac{17}{30} & \frac{23}{20} \end{pmatrix} r - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{23}{20} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{45} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{120} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{150} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{90} \end{pmatrix} \right| = 0, \tag{45}$$

yields

$$\{\{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 1\}\}.$$

For $k = 4$, we have

$$\left| \begin{pmatrix} -\frac{83}{70} & \frac{7}{2} & -\frac{7}{3} & \frac{35}{24} & -\frac{7}{10} & \frac{7}{30} & -\frac{1}{21} & \frac{1}{224} \\ -\frac{363}{245} & \frac{1}{10} & 2 & -\frac{5}{6} & \frac{1}{3} & -\frac{1}{10} & \frac{2}{105} & -\frac{1}{588} \\ -\frac{2}{7} & -\frac{39}{10} & \frac{11}{5} & \frac{5}{2} & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{35} & \frac{1}{420} \\ \frac{1}{21} & -\frac{1}{2} & -\frac{29}{10} & \frac{5}{2} & 1 & -\frac{1}{6} & \frac{1}{42} & -\frac{1}{560} \\ -\frac{2}{105} & \frac{2}{15} & -\frac{4}{5} & -2 & \frac{58}{25} & \frac{2}{5} & -\frac{4}{105} & \frac{1}{420} \\ \frac{1}{70} & -\frac{1}{12} & \frac{1}{3} & -\frac{5}{4} & -\frac{11}{10} & \frac{39}{20} & \frac{1}{7} & -\frac{1}{168} \\ -\frac{2}{105} & \frac{1}{10} & -\frac{1}{3} & \frac{5}{6} & -2 & -\frac{1}{10} & \frac{363}{245} & \frac{1}{28} \\ \frac{1}{21} & -\frac{7}{30} & \frac{7}{10} & -\frac{35}{24} & \frac{7}{3} & -\frac{7}{2} & \frac{83}{70} & \frac{1041}{1120} \end{pmatrix} r - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1041}{1120} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{84} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{336} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{840} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{588} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{224} \end{pmatrix} \right| = 0, \quad (46)$$

which leads to

$$\frac{999435592948462387200000000r^8 - 999435592948462387200000000r^7}{1214591866430423040000000000},$$

and gives
$\{\{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 1\}\}$.

For $k = 5$, we have

$$\left| \begin{pmatrix} -\frac{2089}{1260} & \frac{9}{2} & -4 & \frac{7}{2} & -\frac{63}{25} & \frac{7}{5} & -\frac{4}{7} & \frac{9}{56} & -\frac{1}{36} & \frac{1}{450} \\ -\frac{7129}{5670} & \frac{61}{140} & \frac{8}{3} & -\frac{14}{9} & \frac{14}{15} & -\frac{7}{15} & \frac{8}{45} & -\frac{1}{21} & \frac{1}{126} & -\frac{1}{1620} \\ -\frac{2}{9} & -\frac{621}{140} & \frac{404}{315} & \frac{14}{3} & -\frac{28}{15} & \frac{7}{9} & -\frac{4}{15} & \frac{1}{15} & \frac{2}{189} & \frac{1}{1260} \\ \frac{1}{36} & -\frac{3}{8} & -\frac{739}{210} & \frac{133}{60} & \frac{21}{10} & -\frac{7}{12} & \frac{1}{6} & -\frac{3}{80} & \frac{1}{180} & -\frac{1}{2520} \\ -\frac{1}{126} & \frac{1}{14} & -\frac{4}{7} & -\frac{41}{15} & \frac{12}{5} & 1 & -\frac{4}{21} & \frac{1}{28} & -\frac{1}{210} & \frac{1}{3150} \\ \frac{1}{252} & -\frac{5}{168} & \frac{10}{63} & -\frac{5}{6} & -2 & \frac{41}{18} & \frac{10}{21} & -\frac{5}{84} & \frac{5}{756} & -\frac{1}{2520} \\ -\frac{1}{315} & \frac{3}{140} & -\frac{2}{21} & \frac{1}{3} & -\frac{6}{5} & -\frac{19}{15} & \frac{1478}{735} & \frac{3}{14} & -\frac{1}{63} & \frac{1}{1260} \\ \frac{1}{252} & -\frac{1}{40} & \frac{1}{10} & -\frac{7}{24} & \frac{7}{10} & -\frac{7}{4} & -\frac{101}{210} & \frac{1863}{1120} & \frac{1}{12} & -\frac{1}{360} \\ -\frac{1}{126} & \frac{1}{21} & -\frac{8}{45} & \frac{7}{15} & -\frac{14}{15} & \frac{14}{9} & -\frac{8}{3} & \frac{61}{140} & \frac{7129}{5670} & \frac{1}{45} \\ \frac{1}{36} & -\frac{9}{56} & \frac{4}{7} & -\frac{7}{5} & \frac{63}{25} & -\frac{7}{2} & 4 & -\frac{9}{2} & \frac{2089}{1260} & \frac{9901}{12600} \end{pmatrix} r - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9901}{12600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{135} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{720} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2100} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3780} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4410} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3360} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{126} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} \end{pmatrix} \right| = 0, \quad (47)$$

which leads to

$$\frac{-\zeta_0 r^{10} - \zeta_1 r^9}{\zeta_3},$$

where $\zeta_0 = 388865308064695018170354771614760960000000$, $\zeta_1 = 388865308064695018170354771614760960000000$, and $\zeta_3 = 1646952211897640125217832960000000000000$, and gives
$\{\{r \to 1\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}, \{r \to 0\}\}$.

We observed that the determinant of the first characteristics polynomial of the continous method equation (14) is of the form

$$det(A_1 r - A_0) = r^{2k}(r - 1) \quad \text{for all } k \geq 1. \quad (48)$$

Since the determinant of the first characteristics polynomial of equation (14) can be transformed to the form equation (48) which possesses all roots inside the unit modulus and only one root on the boundary of the unit modulus, then the hybrid schemes of equation (14) are all zero stable for all number of k.

3.2. A-STABILITY

A linear multistep method is said to be an A-stable method if its region of stability encloses the entire negative half-plane Ref. [22].

$$y' = \lambda y, \qquad Re(\lambda) < 0. \quad (49)$$

when applied on equation (14) gives

$$(A_1 - zB_1)Y_{n+1} = (A_0 + zB_0)Y_n, where \quad z = \lambda h. \quad (50)$$

Then simplified into

$$Y_{n+1} = M(z)Y_n, \quad (51)$$

where the $M(z)$ is the amplification matrix given as

$$M(z) = (A_1 - zB_1)^{-1}(A_0 + zB_0). \quad (52)$$

By computing the eigenvalues of the matrix $M(z)$, the stability funtion $P(z)$ is deduced. If in this interval $P(z) < 1$, then it is said to be A-stable. The stability function of the proposed continous scheme for the case of $k = 3, 4$ and 6 are presented as below: For $k = 3$

$$P_3(z) = \Big(27290000z^6 + 114574500z^5 + 734058450z^4 + 2491318620z^3 + 5860559133z^2 + 11021391915z + 6677977600\Big) / \Big(25437400z^6 - 6089580z^5 + 21303162z^4 + 9774243z^3 - 127992888z^2 - 79278246z - 305699840\Big), \quad (53)$$

For $k = 4$

$$P_4(z) = \Big(967680 + 1935360z + 1834560z^2 + 1088640z^3 + 448980z^4 + 134568z^5 + 29531z^6 + 4566z^7 + 420z^8\Big) \Big/ \Big(967680 - 1935360z + 1834560z^2 - 1088640z^3 + 448980z^4 - 134568z^5 + 29531z^6 - 4566z^7 + 420z^8\Big), \quad (54)$$

For $k = 5$

$$P_5(z) = \Big(147027636000000z^{10} + 182325937500000z^{9} + 1219715846811000z^{8} + 7112060709478200z^{7} + 25423388787359640z^{6} + 95242770262622430z^{5} + 189866342878684353z^{4} + 448801094001115262z^{3} + 440792645381904633z^{2} + 508624424469561064z + 144270723334352356\Big) / \Big(15876(926100000z^{10} - 11745300000z^{9} + 91089442500z^{8} - 514248432950z^{7} + 2204475600815z^{6} - 6945780484495z^{5} + 16357302593693z^{4} - 24488766545084z^{3} + 21261955174848z^{2} + 392372936487z - 9839852767962\Big), \tag{55}$$

For $k = 6$

$$P_6(z) = \Big(99843767100000z^{12} + 1372145831595000z^{11} + 9237418277645250z^{10} + 62442484611001275z^{9} + 247886337382504945z^{8} + 1088883507688419366z^{7} + 2829967550818164966z^{6} + 8176692105147411648z^{5} + 13522316566513030848z^{4} + 23638084406015270400z^{3} + 21526034292121075200z^{2} + 17408593681414594560z + 4940882522434805760\Big) / \Big(99843767100000z^{12} - 1239344976870000z^{11} + 9662807742588000z^{10} - 56589936328926000z^{9} + 263019104818470000z^{8} - 989649834673500000z^{7} + 3024911882652660000z^{6} - 7451380276179840000z^{5} + 14512331485491840000z^{4} - 21598203699072000000z^{3} + 23148843964646400000z^{2} - 15949442731622400000z + 5316480910540800000\Big). \tag{56}$$

A region of absolute stability for the method of equation (14) can be defined as

$$Re = |z \in C : M(z) < 1|. \tag{57}$$

The method of equation (14) is said to be A- stable if the region of its absolute stability contains the entire left of the complex plane.

The boundary of the A-stability region obtained through its locus for the continuous scheme is shown in the figure.

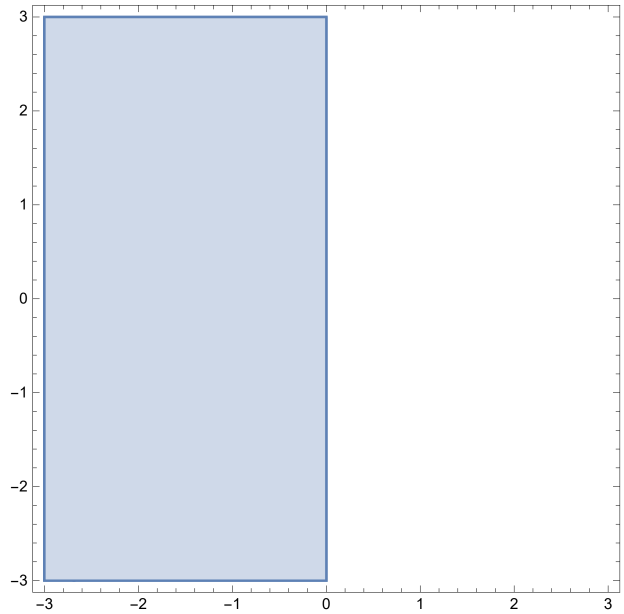

Figure 1. Stability plot for $k = 3$.

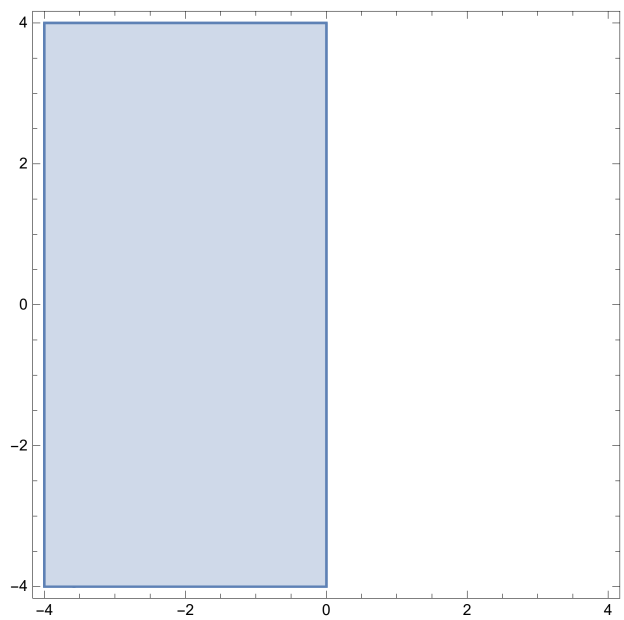

Figure 2. Stability plot for $k = 4$.

4. NUMERICAL EXPERIMENT

The numerical scheme denoted herein is applied on two standard problems in the literature. The non-linearity is reduced using the Newton Rahpson scheme. The Newton-Raphson iterative scheme is considered to resolve the implicitness of equation (14) developed for non-linear problem. Thus block solution $Y_{n+1} = Y_{n+1}^{[q]}$, in equation (14) is iteratively obtained from

$$Y_{n+1}^{[i+1]} = Y_{n+1}^{[i]} - \left(\frac{\partial F(Y_{n+1}^{[i]})}{\partial Y_{n+1}}\right)^{-1} F(Y_{n+1}^{[i]}); \qquad i = 0(1)q \quad q > 1, \tag{58}$$

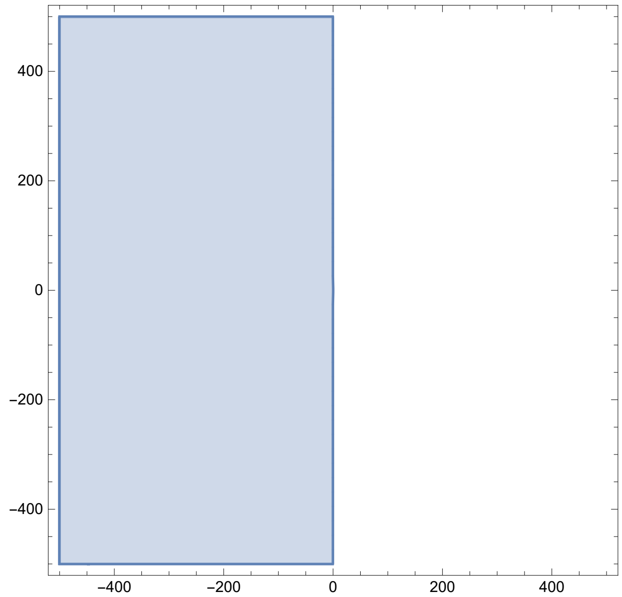

Figure 3. Stability plot for $k = 5$.

where

$$\frac{\partial F(Y_{n+1})}{\partial Y_{n+1}} = \frac{\partial(f_{n+1},\cdots f_{n+N})}{\partial(y_{n+1},\cdots,y_{n+N})} = \begin{pmatrix} \frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{\partial f_{n+1}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+1}}{\partial y_{n+N}} \\ \frac{\partial f_{n+2}}{\partial y_{n+1}} & \frac{\delta f_{n+2}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+2}}{\partial y_{n+N}} \\ \vdots & & & \\ \frac{\partial f_{n+N}}{\partial y_{n+1}} & \frac{\partial f_{n+N}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+N}}{\partial y_{n+N}} \end{pmatrix}. \quad (59)$$

and

$$M(Y_{n+1}) = AY_{n+1} + A_0 Y_n - h(B_0 F_n + BF_{n+1}) = 0. \quad (60)$$

In the Newton's scheme, the criteria for terminating non-linear problems which does not have theoretical solution is $\| Y_{n+1}^{[i+1]} - Y_{n+1}^{[i]} \| < Tol$, where Tol is the accuracy tolerance of the approximations, defined by the user.

This section contains some numerical computations done in Matlab. To show the effectiveness, the application of equation (14) for solving five standard problems are presented herein.

PROBLEM 1:

Consider the problem

This model belongs to the category of epidemiological models, providing valuable insights into the dynamics of infectious diseases within a closed population over a defined time frame. Typically, such models involve a coupled system that accounts for the number of susceptible individuals, denoted as $S(t)$, the number of infected individuals, denoted as $I(t)$, and the number of individuals who have recovered from the disease, denoted as $R(t)$. This foundational model serves as a fundamental framework for understanding the dynamics of various infectious diseases, including but not limited to diseases like measles. The model is given as:

$$S' = \tau(1 - S) - \beta IS, \quad (61)$$

$$I' = I(\tau - \gamma) + \beta IS, \quad (62)$$

$$R' = -\tau R + \gamma I, \quad (63)$$

with β, τ and γ are positive parameters. Suppose, the function $y(t) = S(t) + I(t) + R(t)$. Then the addition of equation (54), equation (55), and equation (56) leads to:

$$y' = \tau(1 - y); \quad \tau = 0.5, \quad y(0) = 0.5, \quad (64)$$

where the theoretical solution is given as $y(t) = 1 - \tau e^{\tau t}$.

Table 1 show the numerical results of equation (14) of order $p = 4$, the method of order $p = 4$ Ref. [19] and Chebyshev method of order $p = 4$ in Ref. [18].

From the computed result in Table 1, it shows that the new scheme of equation (14) of order p= 4 performed much better than those in Ref. [19] of order $p = 4$ and Ref. [18] of order $p = 4$. Hence, the new schemes is more suitable for solving SIR model.

Table 1. Comparison of results for problem 1 , error $y = \mid y - y(t) \mid$.

t	MCHTF p=4 error y	OSHBM p=4 error y	HATM p=4 error y
0.1	1.110×10^{-15}	1.714×10^{-14}	6.780×10^{-13}
0.2	2.220×10^{-15}	3.260×10^{-14}	6.359×10^{-13}
0.3	3.108×10^{-15}	4.653×10^{-14}	6.380×10^{-13}
0.4	4.107×10^{-15}	5.902×10^{-14}	1.189×10^{-12}
0.5	4.551×10^{-15}	7.018×10^{-14}	1.124×10^{-12}
0.6	5.332×10^{-15}	8.011×10^{-14}	1.099×10^{-12}
0.7	6.106×10^{-15}	8.891×10^{-14}	1.547×10^{-12}
0.8	6.439×10^{-15}	9.665×10^{-14}	1.468×10^{-12}
0.9	6.994×10^{-15}	1.034×10^{-13}	1.419×10^{-12}
1.0	7.438×10^{-15}	1.093×10^{-13}	1.782×10^{-12}

Table 2. Comparison of results for problem 2, error $y_i = \mid y_i - y_i(t) \mid$, $i = 1, 2$.

h	Methods	error y_1	error y_2	N	FEVAL	CPU
$\frac{1}{2^5}$	MCHTF5	6.433×10^{-14}	9.448×10^{-14}	32	160	0.096
	IMBlock	4.175×10^{-12}	6.264×10^{-12}	32	160	0.094
$\frac{1}{2^6}$	MCHTF5	1.783×10^{-15}	0	64	320	0.210
	IMBlock	3.997×10^{-15}	5.995×10^{-15}	32	320	0.203
$\frac{1}{2^7}$	MCHTF5	2.482×10^{-16}	1.017×10^{-15}	128	640	0.362
	IMBlock	1.110×10^{-15}	8.887×10^{-15}	128	640	0.359

PROBLEM 2:

Consider the problem

$$y' = \begin{pmatrix} -10 & 6 \\ 13.5 & -10 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} \frac{4e}{3} \\ 0 \end{pmatrix}, \quad (65)$$

$$y(t) = \begin{pmatrix} \frac{2e}{3}\left(e^{-t} + e^{-19t}\right) \\ e\left(e^{-t} - e^{-10t}\right) \end{pmatrix}.$$

The numerical results for various stepsize h are presented in Table 2. It can be seen from Table 2 that the MCHTF is the most efficient method when compared with IMBLOCK in Ref. [16] even at low order. Here FEVAL means number of function evaluation, N is the number of step taken and CPU is the computation time of the CPU.

Table 3. Numerical results for problem 3 , error $y = | y - y(t) |$, $h = 0.01$.

t	MCHTF4	OSHBM
0.01	1.613×10^{-10}	1.558×10^{-6}
0.02	2.140×10^{-10}	2.399×10^{-6}
0.03	2.229×10^{-10}	2.830×10^{-6}
0.04	2.142×10^{-10}	3.020×10^{-6}
0.05	1.990×10^{-10}	3.069×10^{-6}
0.06	1.823×10^{-10}	3.034×10^{-6}
0.07	1.659×10^{-10}	2.951×10^{-6}
0.08	1.508×10^{-10}	2.840×10^{-6}
0.09	1.371×10^{-10}	2.717×10^{-6}
0.10	1.250×10^{-10}	2.588×10^{-6}

Table 4. Numerical results for problem 4, error $y = | y - y(t) |$, $h = 0.1$.

T	MCHTF3 error y	MCHTF5 error y	ODE15s error y
1	5.763×10^{-5}	6.516×10^{-7}	3.981×10^{-4}
2	5.347×10^{-6}	1.945×10^{-7}	2.973×10^{-4}
3	1.246×10^{-6}	1.186×10^{-8}	8.459×10^{-4}
4	1.559×10^{-7}	5.787×10^{-9}	2.265×10^{-4}
5	1.407×10^{-8}	4.934×10^{-10}	2.830×10^{-4}
6	1.121×10^{-9}	4.761×10^{-11}	3.077×10^{-4}
7	8.335×10^{-11}	3.344×10^{-12}	4.234×10^{-4}
8	5.935×10^{-12}	2.620×10^{-13}	1.107×10^{-4}
9	4.107×10^{-13}	1.776×10^{-14}	5.997×10^{-5}
10	2.797×10^{-14}	8.881×10^{-16}	2.237×10^{-4}

PROBLEM 3:

Given the IVPs

$$y\prime = -10(y-1)^2, \quad y(0) = 2, \quad t \in [0, 0.1]\,. \tag{66}$$

The theoretical solution is $y = 1 + \frac{1}{1+10t}$. The MCHTF5 is applied to problem 3 and the error $(| y - y(t) |)$ in the various interval $0 < t \leq 10$ are computed in Table 3. It is clear from the numerical result and comparison in Table 3 that the MCHTFs5 performs better than the methods ($p = 5$) in Ref. [23], the methods (p=5) in Ref. [24] and methods (p=5) of Ref. [18].

PROBLEM 4:

Consider the quadratic Riccati differential equation

$$y\prime = 1 + 2y - y^2, \quad y(0) = 0, \quad t \in [0, 10]\,. \tag{67}$$

The theoretical solution is $y = 1 + \sqrt{2}\tanh\left[\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$. The problem 4 is solved using the MCHTFs for $k = 2, 3$ (MCHTFs3 of order $p = 3$ and MCHTFs5 of order $p = 5$), respectively. The result are given in Table 4 and compared in term of accuracy with the results of ODE15s. Thus the MCHTFs perform better than the compared ODE15s.

Table 5. Comparison of results for problem 5 on $t \in [0, 1]$.

Steps	MCHTFs5 $p = 5$	GAMs5 $p = 5$
20	9.06×10^{-2}	2.25×10^{-1}
40	1.12×10^{-2}	4.41×10^{-2}
80	2.61×10^{-4}	6.49×10^{-3}
160	6.50×10^{-6}	8.86×10^{-4}
320	1.86×10^{-7}	9.88×10^{-5}

The results from ODE15s at $t = 1$ is $3.660087954 \times 10^{-5}$.

PROBLEM 5:

Consider the linear system given in Ref. [1]

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \tag{68}$$

and the exact solution is given as

$$\begin{aligned} y_1(t) &= \tfrac{1}{2}\left(e^{-2t} + e^{-40t}\left(\cos(40t) + \sin(40t)\right)\right), \\ y_2(t) &= \tfrac{1}{2}\left(e^{-2t} - e^{-40t}\left(\cos(40t) + \sin(40t)\right)\right), \\ y_3(t) &= e^{-40t}\left(\cos(40t) - \sin(40t)\right). \end{aligned} \; .$$

The given ODEs in problem 5 has a stiffness ratio $S = 28.5$ and the eigenvalues of the Jacobian matrix are $\lambda_1 = -2$ and $\lambda_{2,3} = -40 + 40i$. Table 5 contains the maximum relative error $\max_{1<i<3} | y_i(t) - y_{i,h} | / (1 + | y_{i,h} |)$ in the range $0 < t \leq 1$ using MCHTF of order $p = 5$. The MCHTFs5 is compared with Generalized Adams Method (GAM) of order $p = 5$ in Ref. [1]. It as observed that the new schemes MCHTFs5 perform better in accuracy than GAMs6 in Ref. [1].

5. CONCLUSION

A new class of hybrid block method has been introduced and derived as Modified Continous Hybrid Block-Type formula. The MCHTF derived are found to be A-stable for order $p \leq 18$. The boundary loci in Figures 1-3 show that the schemes are stable on test problem. Further more, numerical results by implementing MCHTF to some stiff ODEs are obtained, the numerical results showed that the method is effective and accurate when compared with some existing methods. The development of these family of methods have the potential to revolutionize the numerical analysis of differential equations, enabling more accurate, efficient and robust solutions for a wide range of applications. In the future it can also be applied to fields like machine learning,control theory, signal processing weather forecasting, fluid dynamics and quantum mechanics.

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