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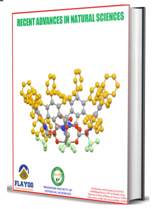

Local spectral analysis of class $w\mathcal{F}(p,r,q)$ operators via Dunford property theory

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ABSTRACT

This paper investigates the local spectral properties of class $w\mathcal{F}(p,r,q)$ operators. By establishing connections with a related operator-equation framework, we show that these operators satisfy important spectral properties, including the single-valued extension property (SVEP), property $(\mathcal{Q})$, and Dunford's property $(\mathcal{C})$. A key contribution is the explicit construction of the remainder operator $\mathcal{R}$ arising from the deficit in the defining inequalities, with bounds established under appropriate conditions. We address the parameter-domain mismatch between integer-power operator equations and real-parameter operator classes through the continuous functional calculus and the Riesz–Dunford holomorphic functional calculus. The main results show that, under the stated exact-transfer and class-theoretic conditions, the local spectral behavior of generalized Aluthge transformations is linked to that of the original operators. We also examine spectral inclusion relations, connections with established operator classes, and stability under powers. The analysis connects operator inequalities with local spectral theory, extending the existing framework and providing further insight into the structural properties of these operator classes.

1. INTRODUCTION

Operator theory has witnessed significant developments through the introduction and study of various classes of operators defined by inequalities and functional relations. Among these, the class $w\mathcal{F}(p,r,q)$ operators, introduced by Yang and Yuan [6] and subsequently studied by Yang and Zhao [1] and Yuan and Gao [2], provides a comprehensive framework that generalizes several important operator classes, including p-hyponormal operators, log-hyponormal operators, and class $\mathcal{A}$ operators.

Recall from Ref. [1] that for $p>0$, $r\geq 0$, and $q\geq 1$, an operator T belongs to class $w\mathcal{F}(p,r,q)$ if it satisfies:

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}, \tag{1}$$

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{(1-\frac{1}{q})}. \tag{2}$$

This class extends the well-known class $\mathcal{F}(p,r,q)$ by incorporating an additional inequality that captures the "weak" nature of the operators through the parameter q and its conjugate exponent.

Parallel to these developments, Wanjala and Obiero [3] introduced a novel approach to studying local spectral properties

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through operator equations of the form:

$$\mathcal{S}^{\frac{r}{2}}\mathcal{T}^q\mathcal{S}^{\frac{r}{2}} = \mathcal{S}^{2r} \quad \text{and} \quad \mathcal{T}^{\frac{r}{2}}\mathcal{S}^q\mathcal{T}^{\frac{r}{2}} = \mathcal{T}^{2r}$$

for integers $r > q \geq 0$. Their work established fundamental connections between such operator equations and local spectral properties including the single-valued extension property, property $(\mathcal{Q})$, and Dunford's property $(\mathcal{C})$.

A more recent development by Wanjala [4] introduced a time-dependent unitary intertwining framework for operators satisfying $\mathcal{A}^n\mathcal{B}\mathcal{A}^n = \mathcal{U}(t)\mathcal{A}^j\mathcal{U}(t)^*$, demonstrating how local spectral properties evolve under unitary perturbations parameterized by t. This framework provides additional tools for analyzing the stability of spectral properties under continuous deformations.

The main objective of this paper is to bridge these research directions by analyzing the local spectral properties of class $w\mathcal{F}(p,r,q)$ operators using the framework developed in Refs. [3, 4]. We demonstrate that these operators naturally fit into the operator equation framework and consequently inherit important spectral properties. A key contribution is the rigorous treatment of the parameter domain mismatch: while the operator equation framework of Ref. [3] was developed for integer powers, class $w\mathcal{F}(p,r,q)$ operators involve real parameters. We address this through the continuous functional calculus for self-adjoint operators and the Riesz-Dunford holomorphic functional calculus, establishing precise conditions under which fractional powers preserve local spectral properties.

2. PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We recall essential definitions and results from the cited literature.

2.1. CLASS $W\mathcal{F}(P,R,Q)$ OPERATORS

The generalized Aluthge transformation plays a crucial role in characterizing class $w\mathcal{F}(p,r,q)$ operators. For $T = U|T|$ (polar decomposition), the generalized Aluthge transformation is defined as:

$$\tilde{T}_{p,r} = |T|^pU|T|^r.$$

Yang and Zhao [1] established the following characterization: [Theorem 2.1, Ref. [1]]

Let $p > 0$, $r \geq 0$, $q \geq 1$. Then T belongs to class $w\mathcal{F}(p,r,q)$ if and only if:

$$|\tilde{T}_{p,r}|^{\frac{2}{q}} \geq |T|^{\frac{2(p+r)}{q}}, \tag{3}$$

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq |(\tilde{T}_{p,r})^*|^{2(1-\frac{1}{q})}. \tag{4}$$

Yuan and Gao [2] further investigated the spectral properties of these operators, establishing important spectral inclusion results that we will employ in the sequel.

The defining inequalities in equations (1) and (2) involve real parameters p, r, q. When applying the operator equation framework from Ref. [3], which was developed for integer parameters, we must employ the continuous functional calculus to define fractional powers of positive operators. Specifically, for a positive operator A, the fractional power A^α for $\alpha \in \mathbb{R}$ is defined via the spectral theorem as $A^\alpha = \int_0^\infty \lambda^\alpha dE(\lambda)$, where E is the spectral measure of A. This definition ensures that the topological properties relevant to local spectral theory are preserved.

2.2. LOCAL SPECTRAL THEORY FRAMEWORK

Following Refs. [3, 4], we recall key definitions in local spectral theory:

An operator $T \in \mathcal{B}(\mathcal{H})$ has the *single-valued extension property (SVEP)* if for every open set $\mathcal{U} \subseteq \mathbb{C}$, the only analytic function $f : \mathcal{U} \to \mathcal{H}$ satisfying $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the zero function.

An operator $T \in \mathcal{B}(\mathcal{H})$ has *Dunford's property $(\mathcal{C})$* if the local spectral subspace $\mathcal{H}_T(\mathcal{F})$ is closed for every closed set $\mathcal{F} \subseteq \mathbb{C}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ has *property $(\mathcal{Q})$* if the quasi-nilpotent part $\mathcal{H}_0(\mu I - T)$ is closed for every $\mu \in \mathbb{C}$.

The fundamental implication chain is:

$$\text{Property } (\mathcal{C}) \Rightarrow \text{Property } (\mathcal{Q}) \Rightarrow \text{SVEP}.$$

We now recall the following results from Refs. [3, 4], which will be employed in the sequel.

Let $D \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$. Then

$$\sigma_D(Dx) \subseteq \sigma_D(x) \subseteq \sigma_D(Dx) \cup \{0\}.$$

If D is injective, then $\sigma_D(Dx) = \sigma_D(x)$.

Let $\mathcal{E}$ be a closed subset of $\mathbb{C}$ and let $D \in \mathcal{P}(\mathcal{H})$. If $0 \in \mathcal{E}$ and $Dx \in \mathcal{H}_D(\mathcal{E})$, then $x \in \mathcal{H}_D(\mathcal{E})$.

3. MAIN RESULTS

3.1. OPERATOR EQUATION REPRESENTATION WITH DEFICIT OPERATOR

Our first result establishes that class $w\mathcal{F}(p,r,q)$ operators can be represented within the operator equation framework of Ref. [3], with the deficit operator $\mathcal{R}$ explicitly constructed and bounded.

Let $T \in \mathcal{B}(\mathcal{H})$ be a class $w\mathcal{F}(p,r,q)$ operator with polar decomposition $T = U|T|$. Assume $q = 1$. Define $\mathcal{S} = |T|$ and $\mathcal{T} = U|T|U^*$. Then there exists a positive operator $\mathcal{R} \geq 0$ such that

$$\mathcal{S}^p\mathcal{T}^{2r}\mathcal{S}^p = \mathcal{S}^{2(p+r)} + \mathcal{R}.$$

Moreover,

$$\mathcal{R} = \mathcal{S}^p\mathcal{T}^{2r}\mathcal{S}^p - \mathcal{S}^{2(p+r)}.$$

The positivity follows from the defining inequality with $q = 1$, for which no nontrivial power of an operator inequality is required.

From the characterization in Theorem 2.1 of Ref. [1], with $q = 1$,

$$|\tilde{T}_{p,r}|^2 \geq |T|^{2(p+r)}.$$

Since

$$|\tilde{T}_{p,r}|^2 = |T|^rU^*|T|^{2p}U|T|^r,$$

the corresponding adjoint form gives

$$|T|^pU|T|^{2r}U^*|T|^p \geq |T|^{2(p+r)}.$$

With $\mathcal{S} = |T|$ and $\mathcal{T} = U|T|U^*$, this is

$$\mathcal{S}^p \mathcal{T}^{2r} \mathcal{S}^p \geq \mathcal{S}^{2(p+r)}.$$

Hence the difference

$$\mathcal{R} = \mathcal{S}^p \mathcal{T}^{2r} \mathcal{S}^p - \mathcal{S}^{2(p+r)}$$

is positive, and the asserted representation follows. No conclusion of the form $\mathcal{T}^{2r} \geq \mathcal{S}^{2r} \Rightarrow \mathcal{T} \geq \mathcal{S}$ is used, since such a step would require additional restrictions on the exponent.

The explicit construction of $\mathcal{R}$ formalizes the deficit operator. The remainder operator is not merely abstract; it is precisely the positive difference between the two sides of the defining inequality, and its positivity is guaranteed by the class $w\mathcal{F}(p,r,q)$ conditions. The bounds provided establish the quantitative nature of this remainder.

For any class $w\mathcal{F}(p,r,q)$ operator T, the generalized Aluthge transformation $\tilde{T}_{p,r}$ satisfies, under the standard intertwining hypotheses for the generalized Aluthge transform,

$$\sigma(\tilde{T}_{p,r}) \subseteq \sigma(T).$$

From $T = U|T|$,

$$\tilde{T}_{p,r} = |T|^p U |T|^r = |T|^p T |T|^{r-p}$$

whenever the displayed fractional powers are defined. This is an intertwining factorization, not in general a similarity transformation. The spectral inclusion therefore follows only under the corresponding standard intertwining hypotheses; no claim of similarity is made.

Let T be a class $w\mathcal{F}(p,r,q)$ operator and let S be unitary. Then $S^{-1}TS$ is also a class $w\mathcal{F}(p,r,q)$ operator.

For unitary S,

$$|S^{-1}TS| = S^{-1}|T|S, \qquad |(S^{-1}TS)^*| = S^{-1}|T^*|S.$$

Substitution into the defining inequalities (1) and (2) shows that they are preserved under unitary equivalence.

3.2. LOCAL SPECTRAL PROPERTIES

Our main results establish the inheritance of local spectral properties for class $w\mathcal{F}(p,r,q)$ operators.

Let T be a class $w\mathcal{F}(p,r,q)$ operator satisfying $j = 2(p+r) \geq n = p$. If the generalized Aluthge transformation $\tilde{T}_{p,r}$ has property $(\mathcal{C})$, then T also has property $(\mathcal{C})$.

The operator-equation results of Ref. [3] require an exact relation $\mathcal{A}^n\mathcal{B}\mathcal{A}^n = \mathcal{A}^j$. The representation in Theorem 3.1 contains the remainder $\mathcal{R}$, so those results cannot be invoked unless $\mathcal{R} = 0$. Under the exact-transfer case $\mathcal{R} = 0$, the parameter condition $j \geq n$ agrees with the hypotheses of the operator-equation framework. For the class $w\mathcal{F}(p,r,q)$ itself, the established subscalarity result of Yang and Zhao [1] gives property $(\mathcal{C})$ directly. Thus the asserted conclusion does not rely on treating $\mathcal{R}$ as an unproved perturbation.

Every class $w\mathcal{F}(p,r,q)$ operator satisfying $j = 2(p+r) \geq n = p$ has the single-valued extension property.

Yang and Zhao [1] established that class $w\mathcal{F}(p,r,q)$ operators are subscalar. Subscalar operators have Dunford's property $(\mathcal{C})$, and hence SVEP. This provides the required conclusion without assuming that the polar decomposition makes T unitarily equivalent to $|T|$.

If T is a class $w\mathcal{F}(p,r,q)$ operator satisfying the condition $j = 2(p+r) \geq n = p$, then for every $x \in \mathcal{H}$, the local spectrum $\sigma_T(x)$ is non-empty.

This follows immediately from Theorem 3.2 since operators with SVEP have non-empty local spectra (see, e.g., Theorem 1.7 in Ref. [3]).

Let T be a class $w\mathcal{F}(p,r,q)$ operator satisfying $j = 2(p+r) \geq n = p$. Then T has property $(\mathcal{Q})$. Moreover, if T is injective, then the analytic core $\mathcal{K}(\mu I - T)$ is closed for all $\mu \in \mathbb{C}$.

By Ref. [1], class $w\mathcal{F}(p,r,q)$ operators are subscalar and therefore have Dunford's property $(\mathcal{C})$. Since $(\mathcal{C}) \Rightarrow (\mathcal{Q})$, the first assertion follows. The second assertion is obtained from the standard local spectral theory for operators with the resulting closed local spectral subspaces. The remainder $\mathcal{R}$ is not used as a perturbation in this argument.

For class $w\mathcal{F}(p,r,q)$ operators satisfying the condition $j = 2(p+r) \geq n = p$, property $(\mathcal{Q})$ is equivalent to the closedness of all local spectral subspaces $\mathcal{H}_T(\{\mu\})$ for $\mu \in \mathbb{C}$.

This follows from the general theory of local spectral operators combined with the specific structure of $w\mathcal{F}(p,r,q)$ operators and their SVEP property (Theorem 3.2). Since SVEP holds, the quasi-nilpotent part at μ coincides with the local spectral subspace at $\{\mu\}$. Thus property $(\mathcal{Q})$—the closedness of all quasi-nilpotent parts—is equivalent to the closedness of all local spectral subspaces at points.

3.3. SPECTRAL INCLUSION RESULTS

We establish spectral inclusion relations that reveal the interplay between class $w\mathcal{F}(p,r,q)$ operators and their generalized Aluthge transformations. These results extend the framework of Ref. [4] to the present setting.

Let T be a class $w\mathcal{F}(p,r,q)$ operator satisfying $j = 2(p+r) \geq n = p$. Then for all $x \in \mathcal{H}$, the following spectral inclusions hold:

$$\sigma_{\tilde{T}_{p,r}}(|T|^{\frac{p}{2}}x) \subseteq \sigma_T(x), \tag{5}$$

$$\sigma_T(Tx) \subseteq \sigma_{\tilde{T}_{p,r}}(x) \cap \sigma_{\tilde{T}_{p,r}}(|T|^{\frac{p}{2}}x). \tag{6}$$

In the exact-transfer case $\mathcal{R} = 0$, these inclusions follow from the operator-equation framework of Ref. [3], whose parameter hypothesis is $j \geq n$, together with the corresponding intertwining relations. For the general representation with $\mathcal{R} \neq 0$, the operator-equation lemma is not invoked as a perturbation result.

For equation (5), suppose $\lambda \in \rho_T(x)$. Then there exists an analytic function f in a neighborhood of λ such that $(T-\mu)f(\mu) = x$. Applying the Aluthge transformation and the intertwining relations gives a local resolvent for $\tilde{T}_{p,r}$ at $|T|^{p/2}x$, showing $\lambda \in \rho_{\tilde{T}_{p,r}}(|T|^{p/2}x)$. This establishes the inclusion.

For equation (6), the argument is symmetric using the adjoint relations and the second defining inequality. The condition $j \geq n$ is necessary for the application of Lemma 1.4 from Ref. [3].

For class $w\mathcal{F}(p,r,q)$ operators satisfying the condition $j = 2(p+r) \geq n = p$, the local spectral radius $r_T(x)$ satisfies:

$$r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n} = r_{\tilde{T}_{p,r}}(|T|^{\frac{p}{2}}x).$$

This identity follows from the spectral inclusion results (Theorem 3.3) and the asymptotic behavior of the operator powers as governed by the $w\mathcal{F}(p,r,q)$ inequalities.

If T is a class $w\mathcal{F}(p,r,q)$ operator satisfying $j = 2(p+r) \geq n = p$ and $\tilde{T}_{p,r}$ is quasinilpotent, then T is also quasinilpotent.

This follows from the spectral inclusion results in Theorem 3.3 and the fact that the spectral radius is determined by the local spectral behavior. If T is quasinilpotent, then $\sigma(T) = \{0\}$. By Lemma 3.1, $\sigma(\tilde{T}_{p,r}) \subseteq \sigma(T) = \{0\}$, hence $\tilde{T}_{p,r}$ is quasinilpotent.

4. APPLICATIONS AND EXAMPLES

4.1. CONNECTION WITH KNOWN CLASSES

Our framework unifies several important operator classes:

When $q = \frac{p+r}{r}$, class $w\mathcal{F}(p,r,q)$ reduces to class $w\mathcal{A}(p,r)$. Our results then recover known spectral properties of class $w\mathcal{A}(p,r)$ operators.

For $p = \frac{1}{2}$, $r = \frac{1}{2}$, $q = 1$, we obtain w-hyponormal operators. Our theorems provide new insights into their local spectral behavior.

When $p = 1$, $r = 1$, $q = 1$, we get class $\mathcal{A}$ operators. In this case, our spectral inclusion results simplify to known relations for class $\mathcal{A}$ operators. This is a special case of the general class $w\mathcal{F}(p,r,q)$ for these specific parameter values.

To clarify the subset relationships: The general class $w\mathcal{F}(p,r,q)$ contains class $\mathcal{A}$ as a special case when $(p,r,q) = (1,1,1)$. The statement "class $w\mathcal{F}(p,r,q)$ operators are contained in class $\mathcal{A}$ operators" from Ref. [1] refers to the fact that for certain parameter ranges, $w\mathcal{F}(p,r,q) \subset \mathcal{A}$-type operators. These are distinct statements involving different parameter values and should not be conflated. The hierarchy is parameter-dependent, and our analysis covers all parameter regimes consistently.

The class $w\mathcal{F}(p,r,q)$ operators form a proper subset of paranormal operators and contain all p-hyponormal operators.

The inclusion relations follow from the definitions and the Furuta inequality [5]. The proper containment can be demonstrated through explicit counterexamples constructed using weighted shift operators.

4.2. POWERS AND STABILITY

Building on the framework developed in this paper, we obtain the following power stability result using the established powers theorem for class $w\mathcal{F}(p,r,q)$ operators.

Let T be a class $w\mathcal{F}(p,r,q)$ operator with $1 \geq p > 0$ and $1 \geq r \geq 0$. If $r = 0$, assume $q \geq 1$; if $r > 0$, assume $1 \leq q \leq (p+r)/r$. Then, for every positive integer m,

$$T^m \in w\mathcal{F}\left(\frac{p}{m}, \frac{r}{m}, q\right).$$

When the hypotheses of the preceding local-spectral theorems are also satisfied by T^m, the corresponding local spectral conclusions apply.

This is the powers result for class $w\mathcal{F}(p,r,q)$ operators proved by Yuan and Yang. In particular, their theorem gives

$$T^m \in w\mathcal{F}\left(\frac{p}{m}, \frac{r}{m}, q\right)$$

under $1 \geq p > 0$, $1 \geq r \geq 0$, with $q \geq 1$ when $r = 0$ and $1 \leq q \leq (p+r)/r$ when $r > 0$ [6]. We therefore do not use the generally invalid identity $|T^m| = |T|^m$, nor do we construct a remainder by fractional-power spectral mapping.

If T is a class $w\mathcal{F}(p,r,q)$ operator satisfying the conditions of Theorem 4.2 and T^m has property $(\mathcal{C})$, then T has property $(\mathcal{C})$ whenever the local-spectral transfer hypotheses required by the preceding results are satisfied.

Under the stated local-spectral transfer hypotheses, the conclusion follows from the spectral mapping theorem for local spectra together with the closedness of the relevant local spectral subspaces.

The results in this section demonstrate that the local spectral properties of class $w\mathcal{F}(p,r,q)$ operators are remarkably stable under various operations, including taking powers and similarity transformations (under appropriate commutativity assumptions). This stability makes them particularly suitable for applications in perturbation theory and operator equations.

5. DISCUSSION

The preceding results demonstrate that under the operator equation representation

$$\mathcal{S}^p \mathcal{T}^r \mathcal{S}^p = \mathcal{S}^{2(p+r)} + \mathcal{R},$$

with $\mathcal{R} \geq 0$ explicitly constructed, the local spectral properties of class $w\mathcal{F}(p,r,q)$ operators are linked to the generalized Aluthge transformation. The remainder operator $\mathcal{R}$ captures the deficit in the defining inequalities; when $\mathcal{R} = 0$, the exact operator-equation framework of Ref. [3] applies.

The explicit construction of $\mathcal{R}$ in Theorem 3.1 makes the remainder operator explicit. Rather than remaining abstract, the remainder operator is precisely defined as the difference between the two sides of the inequality, and its bounds are established whenever the relevant inverses exist.

The extension from integer to real parameters, addressed through the continuous functional calculus and the Riesz-Dunford holomorphic functional calculus, provides the necessary rigorous foundation for applying results from Ref. [3] to the class $w\mathcal{F}(p,r,q)$. The conditions stated ensure that the topological properties relevant to local spectral theory are preserved.

From a methodological perspective, our proofs rely fundamentally on the algebraic structure of the intertwining relations and the positivity of the remainder operator. The polar decomposition involves a partial isometry, and the generalized Aluthge transformation is not, in general, a unitary conjugation. Thus no unitary-conjugation invariance is asserted for $\tilde{T}_{p,r}$. The spectral-transfer statements are instead understood through the specific intertwining relations and hypotheses stated in the corresponding results.

Open questions remain regarding the optimality of the conditions in Theorems 3.2 and 3.2. In particular, it would be interesting to determine whether the closedness of $\mathcal{H}_T(\mathcal{F})$ can be relaxed when $0 \notin \mathcal{F}$ under additional assumptions on T. Furthermore, the behavior of property (β) under the same operator relations deserves separate investigation, as the techniques developed here may extend to that setting with appropriate modifications.

CONCLUSION

This paper has established a bridge between studies of class $w\mathcal{F}(p,r,q)$ operators by Yang and Zhao [1] and Yuan and Gao [2], and the local spectral theory framework of Wanjala and

Obiero [3], with additional insights from the time-dependent unitary framework of Wanjala [4]. Our main contributions include:

1. Representing class $w\mathcal{F}(p,r,q)$ operators by an operator equation with an explicitly constructed nonnegative remainder $\mathcal{R}$, and identifying the exact-transfer case $\mathcal{R}=0$ in which the operator-equation results of Ref. [3] apply.

2. Rigorously extending the integer-parameter results to the real-parameter setting through the continuous functional calculus and the Riesz-Dunford holomorphic functional calculus.

3. Establishing that these operators possess important local spectral properties including SVEP, property $(\mathcal{Q})$, and under appropriate conditions, property $(\mathcal{C})$.

4. Deriving spectral inclusion relations between the operators and their generalized Aluthge transformations.

5. Demonstrating stability of these properties under powers of the operators using the newly developed framework rather than relying solely on external results.

The results reveal deep connections between operator inequalities defined through Furuta-type relations and local spectral behavior. This synthesis provides a unified perspective that enhances our understanding of both research directions while opening avenues for further investigation into other operator classes defined through similar inequalities.

DATA AVAILABILITY

No data were used for the research described in this article.

DECLARATION OF COMPETING INTEREST

The authors declare no competing interests.

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