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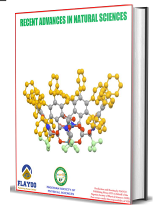

Implicit one-step optimized fourth-derivative hybrid block method for directly solving general third-order IVPs

S. D. Yakubu[a,b,*], P. Sibanda[b]

[a]*Department of Mathematics, Ibrahim Badamasi Babangida University, Lapai, Nigeria*
[b]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Pietermaritzburg 3209, South Africa*

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ABSTRACT

This paper introduces a single-step optimized fourth-derivative block hybrid method specifically designed to solve general third-order initial value problems directly. By incorporating advanced optimization techniques, the method significantly improves accuracy and computational efficiency. Extensive analysis confirms that the method exhibits zero-stability, consistency, A-stability, and convergence properties. Numerical experiments conducted in this study reveal that the proposed method surpasses existing approaches in accuracy, establishing it as a significant advancement in the numerical solution of higher-order initial value problems.

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1. INTRODUCTION

Third-order initial value problems (IVPs) frequently arise in various scientific and engineering applications, including the modeling of electromagnetic waves and thin-film flows. Despite their prevalence, such problems often lack closed-form analytical solutions, necessitating the use of numerical approximations. Traditional methods for solving third-order IVPs often involve transforming them into equivalent systems of first-order ordinary differential equations (ODEs). Although this approach is effective, it significantly increases the system's dimensionality, resulting in higher computational costs and added complexity. To address these challenges, recent studies have prioritized the development of direct numerical techniques for solving third-order IVPs (1), bypassing the need for such transformations.

Several notable contributions have advanced this field. Mohammed and Adeniyi [1] developed a three-step hybrid linear multistep method for solving third-order IVPs. Similarly, Lawal *et al.* [2] introduced a four-step hybrid block technique for special third-order IVPs, achieving an order of $p = 9$ by incorporating four off-step points. Allogmany and Ismail [3] extended the research by deriving a multi-derivative linear multistep method with practical applications for third-order IVPs.

Recent advancements include an implicit one-step hybrid backward differentiation formula by Adamu *et al.* [4], which incorporates four equally spaced off-grid points to directly solve third-order ODEs. Alkasassbeh *et al.* [5] proposed a single-step hybrid block technique incorporating two off-grid points applied to the first and second derivative functions for third-order IVPs. Duromola *et al.* [6] derived a one-step block hybrid method with three equally spaced off-step points for solving third-order IVPs. Adeyefa *et al.* [7] introduced an order-six one-step hybrid block method with a single off-step point for solving third-order ODEs. Kuboye *et al.* [8] developed a four-step family of two hybrid block methods, each incorporating a single off-step point per block, for solving third-order ODEs. Additionally, Rufai *et*

*Corresponding author: Tel.: +234-806-8107-232;
e-mail: saiduyakubu2014@gmail.com (S. D. Yakubu)

al. [9] proposed two-step optimized hybrid technique employing two off-grid points for addressing general third-order ODEs.
Most methods in the literature lack both high-order accuracy and stability. These limitations have driven ongoing efforts to improve the accuracy and efficiency of numerical techniques for directly solving higher-order IVPs. Building on this progress, the present study aims to develop a single-step optimized hybrid block method that leverages three carefully selected off-step points to enhance both stability and accuracy in solving general third-order IVPs, as defined in eq. (1).
The primary objective is to overcome the shortcomings of existing approaches and provide a more efficient and reliable solution for directly addressing third-order IVPs. The robustness of the proposed method is demonstrated through a comprehensive analysis of key properties, including convergence. Additionally, numerical experiments are conducted to evaluate its performance, highlighting its superior accuracy and effectiveness compared to existing methods.
This paper is organized as follows: Section 2 presents the derivation process of the proposed method. Section 3 comprehensively analyzes the method's properties, covering zero-stability, consistency, and convergence. Section 4 presents the numerical results obtained from applying the method and provides a detailed discussion of these results. Finally, Section 5 summarizes the conclusions drawn from the study, highlighting the contributions and implications of the developed method in solving third-order IVPs.

2. DERIVATION OF THE METHOD

This study focuses on developing a numerical method to solve general third-order IVPs, expressed as

$$z'''(t) = f(t, z, z', z''),\ z(t_0) = z_0,\\ z'(t_0) = z_1,\ z''(t_0) = z_2,\ t \in [a, b], \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^{3m} \to \mathbb{R}^m$ is a continuous function that satisfies the Lipschitz condition.

Lemma 1 (Lipschitz condition)**.** *Suppose the third-order ODE is given as eq. (1). The function $f(t, z, z', z'')$ satisfies a Lipschitz condition with respect to z, z', and z'' if there exists a constant $K > 0$ such that, for all points (t, z_1, l_1, ω_1) and (t, z_2, l_2, ω_2) in a domain $D \subset \mathbb{R}^4$, the following inequality holds:*

$$|f(t, z_1, l_1, \omega_1) - f(t, z_2, l_2, \omega_2)| \leq \\ \mathbf{K}(|z_1 - z_2| + |l_1 - l_2| + |\omega_1 - \omega_2|). \tag{2}$$

Here:

- *z_1, z_2 are distinct values for z,*
- *l_1, l_2 are distinct values for z', and*
- *ω_1, ω_2 are distinct values for z''.*

Theorem 1 (Existence and uniqueness of solution)**.** *Suppose the third-order ODE is given as eq. (1). If:*

1. *$f(t, z, z', z'')$ is continuous in t, z, z', z'' in some region $D \subset \mathbb{R}^4$ containing (t_0, z_0, z_1, z_2), and*
2. *$f(t, z, z', z'')$ satisfies the Lipschitz condition in z, z', z'', as defined in eq. (2),*

then there exists a unique solution $z(t)$ to the ODE in some interval $[t_0 - \delta, t_0 + \delta]$, where $\delta > 0$ depends on f and the initial conditions.

Proof
Step 1: Reduction to a first-order system.
To reduce the third-order ODE to a system of first-order ODEs, we introduce new variables:

$$l_1 = z, \quad l_2 = z', \quad l_3 = z''. \tag{3}$$

Then:

$$l_1' = l_2, \quad l_2' = l_3, \quad l_3' = f(t, l_1, l_2, l_3). \tag{4}$$

This system can be written in vector form as

$$\mathbf{L}' = \mathbf{F}(t, \mathbf{L}), \tag{5}$$

where:

$$\mathbf{L} = \left[l_1,\ l_2,\ l_3\right], \quad \mathbf{F}(t, \mathbf{L}) = \left[l_2,\ l_3,\ f(t, l_1, l_2, l_3)\right].$$

Step 2: Application of the Picard-Lindelöf theorem [10].
The Picard-Lindelöf theorem ensures the existence and uniqueness of solutions to the system

$$\mathbf{L}' = \mathbf{F}(t, \mathbf{L}), \tag{6}$$

provided that:

1. $\mathbf{F}(t, \mathbf{L})$ is continuous in t and $\mathbf{L}$, and
2. $\mathbf{F}(t, \mathbf{L})$ satisfies the Lipschitz condition in $\mathbf{L}$.

By assumption, $f(t, z, z', z'')$ satisfies these conditions, which implies $\mathbf{F}(t, \mathbf{L})$ does as well.
Step 3: Solution to the original ODE.
The solution to the system is

$$\mathbf{L}(t) = \left[l_1(t),\ l_2(t),\ l_3(t)\right], \tag{7}$$

which exists and is unique on some interval $[t_0 - \delta, t_0 + \delta]$. The first component, $\mathbf{L}(t)$, $l_1(t) = z(t)$, is the unique solution to the original third-order ODE.
To derive the implicit single-step optimized fourth-derivative hybrid block method, we introduce an approximating polynomial $p(t)$, expressed as

$$z(t) \approx p(t) = \sum_{j=0}^{9} a_j t^j, \tag{8}$$

where $a_j \in \mathbb{R}$ are unknown coefficients to be determined by applying interpolation and collocation conditions at carefully selected points. The expressions for the third and fourth derivatives of $p(t)$ are given by

$$z'''(t) = \sum_{j=3}^{9} j(j-1)(j-2) a_j t^{j-3}, \tag{9}$$

$$z''''(t) = \sum_{j=4}^{9} j(j-1)(j-2)(j-3)a_j t^{j-4}. \tag{10}$$

These expressions form the foundation for deriving the hybrid method by applying interpolation and collocation conditions. Specifically, the following conditions are imposed. eq. (8) is interpolated at $t = t_{n+i}, i = 0, \frac{1}{2}, 1$, while eq. (9) is collocated at $t = t_{n+i}, i = 0, r, \frac{1}{2}, 1-r, 1$. Additionally, eq. (10) is collocated at $t = t_{n+i}, i = 0, 1$. Here $0 < p_i < 1$, with p_i chosen as $r, \frac{1}{2}, 1-r$. These conditions yield a system of ten equations corresponding to the interpolation and collocation constraints. Solving this system provides the ten unknown coefficients a_j of the polynomial $p(t)$. The imposed conditions can be summarized as follows:

$$\begin{aligned} p(t_{n+i}) &= z(t_{n+i}), \quad i = 0, \frac{1}{2}, 1, \\ p'''(t_{n+i}) &= f(t_{n+i}), \quad i = 0, r, \frac{1}{2}, 1-r, 1, \\ p''''(t_{n+i}) &= g(t_{n+i}), \quad i = 0, 1. \end{aligned}$$

Substituting the obtained coefficients into eq. (8) and simplifying yields the general method equation

$$z(t) = \sum_{j=0,\frac{1}{2},1} \mu_j(t) z_{n+j} + h^3 \sum_{j=0,r,\frac{1}{2},1-r,1} \zeta_j(t) f_{n+j} + h^4 \sum_{j=0,1} \xi_j(t) g_{n+j}, \tag{11}$$

where n denotes the grid index, $n = 0, 1, 2, ..., N-1$, and the step size h is defined as $h = t_{n+1} - t_n$ within the interval $[a, b]$.

To address general third-order IVPs, additional method equations are derived from the first and second derivatives of eq. (11), expressed as:

$$z'(t) = \frac{1}{h}\left[\sum_{j=0,\frac{1}{2},1} \mu_j'(t) z_{n+j} + h^3 \sum_{j=0,\frac{1}{2},1,-r,1} \zeta_j'(t) f_{n+j} + h^4 \sum_{j=0,1} \zeta_j'(t) g_{n+j}\right], \tag{12}$$

$$z''(t) = \frac{1}{h^2}\left[\sum_{j=0,\frac{1}{2},1} \mu_j''(t) z_{n+j} + h^3 \sum_{j=0,\frac{1}{2},1,-r,1} \zeta_j''(t) f_{n+j} + h^4 \sum_{j=0,1} \zeta_j''(t) g_{n+j}\right]. \tag{13}$$

To ensure accuracy, the following conditions are imposed:

$$z'(t) = \delta(t), \tag{14}$$

and

$$z''(t) = \phi(t). \tag{15}$$

Evaluating eq. (11) at $t = t_{n+j}, j = r, 1-r$, and eqs. (12) and (13) at all points $t = t_{n+j}, j = 0, r, \frac{1}{2}, 1-r, 1$, produces the following general continuous schemes.

$$\begin{aligned} z_{n+r} &= (\frac{-((h^3(1-2r)^2(151+2r(206+r(-4707+2r(7189}{322560(-1+r)r(-1+2r)} \\ &\frac{+2r(-3813+2r(31+2r(705+2r(-251+56r))))))))}{322560(-1+r)r(-1+2r)})f_n \\ &+(\frac{(151h^3-192h^3r-320h^3r^2-576h^3r^3+}{322560(-1+r)r(-1+2r)} \\ &\frac{4288h^3r^4-4800h^3r^5+1600h^3r^6)}{322560(-1+r)r(-1+2r)})f_{n+r}+ \\ &\frac{4h^3(-1+r)r(4+(-1+r)r(15+(-1+r)r}{315(-1+2r)} \\ &\frac{(-8+7(-1+r)r)))}{315(-1+2r)}f_{n+\frac{1}{2}} \\ &+(\frac{-23h^3+192h^3r+64h^3r^2-192h^3r^3-}{322560(-1+r)r(-1+2r)} \\ &\frac{704h^3r^4+960h^3r^5-320h^3r^6}{322560(-1+r)r(-1+2r)})f_{n+1-r}+ \\ &(\frac{23h^3-192h^3r+294h^3r^2+1756h^3r^3-5264h^3r^4+}{322560(-1+r)r(-1+2r)} \\ &\frac{736h^3r^5-3392h^3r^6+51328h^3r^7-95872h^3r^8+}{322560(-1+r)r(-1+2r)} \\ &\frac{64768h^3r^9-14336h^3r^{10}}{322560(-1+r)r(-1+2r)})f_{n+1}+ \\ &(\frac{(-151h^4r+1661h^4r^2-6798h^4r^3+11352h^4r^4+}{322560(-1+r)r(-1+2r)} \\ &\frac{1680h^4r^5-37632h^4r^6 51328h^3r^7-95872h^3r^8+}{322560(-1+r)r(-1+2r)} \\ &\frac{64768h^3r^9-14336h^3r^{10}}{322560(-1+r)r(-1+2r)})g_n+ \\ &(\frac{(-23h^4r+253h^4r^2-1038h^4r^3+1752h^4r^4-}{322560(-1+r)r(-1+2r)} \\ &\frac{1008h^4r^5+2688h^4r^6-12672h^4r^7+20928h^4r^8}{322560(-1+r)r(-1+2r)} \\ &\frac{-14464h^4r^9+3584h^4r^{10}}{322560(-1+r)r(-1+2r)})g_{n+1}+ \\ &(\frac{(322560r-1935360r^2+4193280r^3-3870720r^4+}{322560(-1+r)r(-1+2r)} \\ &\frac{1290240r^5)}{322560(-1+r)r(-1+2r)})z_n+ \\ &\frac{(1290240r^2-5160960r^3+6451200r^4-2580480r^5)}{322560(-1+r)r(-1+2r)}z_{n+\frac{1}{2}} \\ &+\frac{(1-2r)^2r}{-1+2r}z_{n+1}, \end{aligned} \tag{16}$$

$$\begin{aligned} z_{n+1-r} &= (\frac{(h^3(1-2r)^2(-23+2r(50+r(99+2r(-341+}{322560(-1+r)r(-1+2r)} \\ &\frac{2r(-123+2r(49+2r(327+2r(-197+56r)))))))}{322560(-1+r)r(-1+2r)})f_n \\ &+(\frac{(23h^3-192h^3r-64h^3r^2+192h^3r^3+704h^3r^4}{322560(-1+r)r(-1+2r)} \\ &\frac{-960h^3r^5+320h^3r^6}{322560(-1+r)r(-1+2r)})f_{n+r} \end{aligned}$$

$$
\begin{aligned}
&-(\frac{4h^3(-1+r)r(4+(-1+r)r(15+(-1+r)r(-8+}{315(-1+2r)}\\
&\frac{7(-1+r)r)))}{315(-1+2r)})f_{n+\frac{1}{2}}+\\
&(\frac{(-151h^3+192h^3r+320h^3r^2+576h^3r^3-}{322560(-1+r)r(-1+2r)}\\
&\frac{4288h^3r^4+4800h^3r^5-1600h^3r^6}{322560(-1+r)r(-1+2r)})f_{n+1-r}\\
&+(\frac{151h^3-192h^3r-10458h^3r^2+68060h^3r^3-}{322560(-1+r)r(-1+2r)}\\
&\frac{183184h^3r^4+237536h^3r^5-101440h^3r^6-104320h^3r^7}{322560(-1+r)r(-1+2r)}\\
&\frac{+158080h^3r^8-78592h^3r^9+14336h^3r^{10}}{322560(-1+r)r(-1+2r)})f_{n+1}+\\
&(\frac{-23h^4r+253h^4r^2-1038h^4r^3+1752h^4r^4-1008h^4r^5+}{322560(-1+r)r(-1+2r)}\\
&\frac{2688h^4r^6-12672h^4r^7+20928h^4r^8-14464h^4r^9+3584h^4r^{10}}{322560(-1+r)r(-1+2r)})g_n\\
&+(\frac{-151h^4r+1661h^4r^2-6798h^4r^3+11352h^4r^4+1680h^4r^5-}{322560(-1+r)r(-1+2r)}\\
&\frac{37632h^4r^6+64128h^4r^7-52032h^4r^8+21376h^4r^9-3584h^4r^{10}}{322560(-1+r)r(-1+2r)})g_{n+1}\\
&+\frac{-322560r^2+1612800r^3-2580480r^4+1290240r^5}{322560(-1+r)r(-1+2r)}z_n+\\
&\frac{1290240r^2-5160960r^3+6451200r^4-2580480r^5}{322560(-1+r)r(-1+2r)}z_{n+\frac{1}{2}}+\\
&\frac{(1-3r+2r^2)^2}{(-1+r)(-1+2r)}z_{n+1},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
h\delta_n \quad &= (\frac{h^3(1-2r)^2(-151+2(-1+r)r(215+4296(-1+r)r))}{322560r^2(1-3r+2r^2)^2})f_n\\
&+\frac{-(151h^3-174h^3r)}{322560r^2(1-3r+2r^2)^2}f_{n+r}\\
&+\frac{8h^3(-1+r)^2(2+9(-1+r)r)}{315(1-3r+2r^2)^2}f_{n+\frac{1}{2}}+\\
&\frac{(-23h^3+174h^3r)}{322560r^2(1-3r+2r^2)^2}f_{n+1-r}+\\
&(\frac{(23h^3-174h^3r+358h^3r^2+208h^3r^3-}{322560r^2(1-3r+2r^2)^2})\\
&\frac{1544h^3r^4+1728h^3r^5-576h^3r^6}{322560r^2(1-3r+2r^2)^2})f_{n+1}+\\
&(\frac{-151h^4r+1643h^4r^2-6536h^4r^3+12148}{322560r^2(1-3r+2r^2)^2}\\
&\frac{h^4r^4-10656h^4r^5+3552h^4r^6}{322560r^2(1-3r+2r^2)^2})g_n+\\
&\frac{-23h^4r+235h^4r^2-904h^4r^3+1652h^4r^4-1440h^4r^5+480h^4r^6}{322560r^2(1-3r+2r^2)^2}g_{n+1}\\
&+(\frac{-967680r^2+5806080r^3-12579840r^4+11612160r^5-}{322560r^2(1-3r+2r^2)^2}\\
&+\frac{3870720r^6}{322560r^2(1-3r+2r^2)^2})z_n + (\frac{1290240r^2-7741440r^3+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{+16773120r^4-15482880r^5+5160960r^6}{322560r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}} - z_{n+1},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
h\delta_{n+r} \quad &= (\frac{-((h^3(1-2r)^2(151+2r(-256+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{r(-5837+6r(8205+4r(-6255+2r(4179+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{16r(-70+r(-224+r(287+r(-141+26r)))}{322560r^2(1-3r+2r^2)^2})f_n\\
&+(\frac{151h^3-1116h^3r+1116h^3r^2+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{26880h^3r^4-107520h^3r^5+161280h^3r^6}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-107520h^3r^7+26880h^3r^8}{322560r^2(1-3r+2r^2)^2})f_{n+r}+\\
&(\frac{8h^3(-1+r)^2(2+3r(-6+}{315(1-3r+2r^2)^2}\\
&\frac{r(17-14r+r^3(-28+r(70+13(-4+r)r)}{315(1-3r+2r^2)^2})f_{n+\frac{1}{2}}\\
&+(\frac{-23h^3+348h^3r-1116h^3r^2+10752h^3r^5}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-21504h^3r^6+15360h^3r^7-3840h^3r^8}{322560r^2(1-3r+2r^2)^2})f_{n+1-r}\\
&+(\frac{23h^3-348h^3r+1474h^3r^2+772h^3r^3}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-18344h^3r^4+32976h^3r^5-19392h^3r^6+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-1087488h^3r^8+2050560h^3r^9-1926144h^3r^{10}}{322560r^2(1-3r+2r^2)^2}\\
&\frac{+890880h^3r^{11}-159744h^3r^{12}}{322560r^2(1-3r+2r^2)^2})f_{n+1}+\\
&(\frac{-151h^4r+2585h^4r^2-+57988h^4r^4}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-70536h^4r^5-134880h^4r^6+684096h^4r^7-}{322560r^2(1-3r+2r^2)^2}\\
&\frac{1257984h^4r^8+1305600h^4r^9-802560h^4r^{10}}{322560r^2(1-3r+2r^2)^2}\\
&\frac{+273408h^4r^{11}-39936h^4r^{12}}{322560r^2(1-3r+2r^2)^2})g_n+\\
&(\frac{-23h^4r+409h^4r^2-2782h^4r^3+9092h^4r^4}{322560r^2(1-3r+2r^2)^2}\\
&\frac{-15240h^4r^5+23328h^4r^6-90048h^4r^7}{322560r^2(1-3r+2r^2)^2}\\
&\frac{278784h^4r^8-468480h^4r^9+430848h^4r^{10}}{322560r^2(1-3r+2r^2)^2}\\
&(\frac{-205824h^4r^{11}+39936h^4r^{12}}{322560r^2(1-3r+2r^2)^2})g_{n+1}+\\
&(\frac{-967680r^2+7096320r^3-20321280r^4}{322560r^2(1-3r+2r^2)^2}\\
&\frac{+28385280r^5-19353600r^6+}{322560r^2(1-3r+2r^2)^2}\\
&\frac{5160960r^7}{322560r^2(1-3r+2r^2)^2})z_n+\\
&(\frac{1290240r^2-10321920r^3+32256000r^4-}{322560r^2(1-3r+2r^2)^2}\\
&\frac{49029120r^5+36126720r^6-10321920r^7}{322560r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}}\\
&+(-1+4r)z_{n+1},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
h\delta_{n+\frac{1}{2}} = {} & \frac{-h^3(1-2r)^2(-23+4(-1+r)r(23+288(-1+r)rf_n}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-23h^3f_{n+r}}{322560r^2(1-3r+2r^2)^2} - (\frac{h^3(-1+r)^2(325+}{10080(1-3r+2r^2)^2} \\
& \frac{1392(-1+r)r}{10080(1-3r+2r^2)^2})f_{n+\frac{1}{2}} \\
& -\frac{23h^3f_{n+1-r}}{322560r^2(1-3r+2r^2)^2} - (\frac{-23h^3+1520h^3r^2-}{322560r^2(1-3r+2r^2)^2} \\
& \frac{7648h^3r^3+15344h^3r^4-13824h^3r^5+4608h^3r^6}{322560r^2(1-3r+2r^2)^2})f_{n+1} \\
& -(\frac{-23h^4r+235h^4r^2-904h^4r^3+1652h^4r^4-1440h^4r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+480h^4r^6}{322560r^2(1-3r+2r^2)^2})g_n - \\
& (\frac{23h^4r-235h^4r^2+904h^4r^3-1652h^4r^4+1440h^4r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-480h^4r^6}{322560r^2(1-3r+2r^2)^2})g_{n+1} - \\
& (\frac{(322560r^2-1935360r^3+4193280r^4-3870720r^5+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{1290240r^6}{322560r^2(1-3r+2r^2)^2})z_n + z_{n+1},
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
h\delta_{n+1-r} = {} & (\frac{(h^3(1-2r)^2(-23+2r(128+r(-179+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{r(6r(-269+4r(143+2r(77+16r(14-116+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{r(188+r(-119+26r))))))))))}{322560r^2(1-3r+2r^2)^2})f_n + \\
& (\frac{-23h^3+348h^3r-1116h^3r^2+10752h^3r^5-}{322560r^2(1-3r+2r^2)^2} \\
& \frac{21504h^3r^6+15360h^3r^7-3840h^3r^8}{322560r^2(1-3r+2r^2)^2})f_{n+r} \\
& (\frac{8h^3(-1+r)^2(2+3r(-6+r(17-14r+r^3(-28+}{315(1-3r+2r^2)^2} \\
& \frac{r(70+13(-4+r))))}{315(1-3r+2r^2)^2})f_{n+\frac{1}{2}} + \\
& (\frac{151h^3-1116h^3r+1116h^3r^2+26880h^3r^4}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-107520h^3r^5+161280h^3r^6-107520h^3r^7+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{26880h^3r^8}{322560r^2(1-3r+2r^2)^2})f_{n+1-r} + \\
& (\frac{-151h^3+1116h^3r+9022h^3r^2-143108h^3r^3+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{740776h^3r^4-1995984h^3r^5-1690752h^3r^7-}{322560r^2(1-3r+2r^2)^2} \\
& \frac{1387008h^3r^8+3356160h^3r^9-2669568h^3r^{10}+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{1026048h^3r^{11}+2913216h^3r^6-159744h^3r^{12}}{322560r^2(1-3r+2r^2)^2})f_{n+1} + \\
& (\frac{23h^4r-409h^4r^2+2782h^4r^3-9092h^4r^4+15240h^4r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-23328h^4r^6+90048h^4r^7-278784h^4r^8+468480h^4r^9}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-430848h^4r^10+205824h^4r^11-39936h^4r^12}{322560r^2(1-3r+2r^2)^2})g_n + \\
& (\frac{151h^4r-2585h^4r^2+17630h^4r^3-57988h^4r^4+70536h^4r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+134880h^4r^6-684096h^4r^7+1257984h^4r^8-1305600h^4r^9}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+802560h^4r^10-273408h^4r^11+39936h^4r^12}{322560r^2(1-3r+2r^2)^2})g_{n+1} + \\
& (\frac{322560r^2-3225600r^3+11934720r^4-20643840r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+16773120r^6-5160960r^7}{322560r^2(1-3r+2r^2)^2})z_n + (\frac{-1290240r^2+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{10321920r^3-32256000r^4+49029120r^5-36126720r^6}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+10321920r^7}{322560r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}} - (-3+4r)z_{n+1},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
h\delta_{n+1} = {} & \frac{h^3(1-2r)^2(-23+2(-1+r)r(-41}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+72(-1+r)r)}{322560r^2(1-3r+2r^2)^2}f_n + \\
& \frac{+(-23h^3+174h^3r)f_{n+r}}{322560r^2(1-3r+2r^2)^2} + \\
& \frac{8h^3(-1+r)^2(2+9(-1+r)r)f_{n+\frac{1}{2}}}{315(1-3r+2r^2)^2} \\
& (\frac{(151h^3-174h^3r)f_{n+1-r}}{322560r^2(1-3r+2r^2)^2} + \\
& \frac{-151h^3+174h^3r+10138h^3r^2}{322560r^2(1-3r+2r^2)^2} \\
& \frac{-54992h^3r^3+113416h^3r^4-103104h^3r^5+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{34368h^3r^6}{322560r^2(1-3r+2r^2)^2})f_{n+1} + \\
& (\frac{23h^4r-235h^4r^2+904h^4r^3-1652h^4r^4+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{1440h^4r^5-480h^4r^6}{322560r^2(1-3r+2r^2)^2})g_n + \\
& (\frac{151h^4r-1643h^4r^2+6536h^4r^3-12148h^4r^4+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{10656h^4r^5-3552h^4r^6}{322560r^2(1-3r+2r^2)^2})g_{n+1} + \\
& (\frac{22560r^2-1935360r^3+4193280r^4-3870720r^5}{322560r^2(1-3r+2r^2)^2} \\
& \frac{+1290240r^6}{322560r^2(1-3r+2r^2)^2})z_n + \\
& (\frac{-1290240r^2+7741440r^3-16773120r^4+}{322560r^2(1-3r+2r^2)^2} \\
& \frac{15482880r^5-5160960r^6}{322560r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}} + 3z_{n+1},
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
h^2\phi_n \quad &= -\frac{h^3(1-2r)^2(-157+26(-1+r)r(17+504(-1+r)r))}{53760r^2(1-3r+2r^2)^2}f_n \\
&+\frac{(-157h^3+186h^3r)f_{n+r}}{53760r^2(1-3r+2r^2)^2}- \\
&\frac{(8h^3(-1+r)^2(3+14(-1+r)r))f_{n+\frac{1}{2}}}{105(1-3r+2r^2)^2} \\
&+\frac{(29h^3-186h^3r)f_{n+1-r}}{53760r^2(1-3r+2r^2)^2}+(\frac{-29h^3+186h^3r+94h^3r^2}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-2800h^3r^3+7000h^3r^4-6720h^3r^5+2240h^3r^6}{53760r^2(1-3r+2r^2)^2})f_{n+1}+ \\
&(\frac{157h^4r-1849h^4r^2+7640h^4r^3-14460h^4r^4+12768h^4r^5}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-4256h^4r^6}{53760r^2(1-3r+2r^2)^2})g_n+(\frac{29h^4r-313h^4r^2+1240h^4r^3}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-2300h^4r^4+2016h^4r^5-672h^4r^6}{53760r^2(1-3r+2r^2)^2})g_{n+1}+ \\
&\frac{215040r^2-1290240r^3+2795520r^4-2580480r^5+860160r^6}{53760r^2(1-3r+2r^2)^2}z_n \\
&+(\frac{-430080r^2+2580480r^3-5591040r^4+5160960r^5}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-1720320r^6}{53760r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}}+4z_{n+1},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
h^2\phi_{n+r} \quad &= -(\frac{h^3(-157+2r(-221+r(6773+8r(-3878+}{53760(-1+r)^2r^2} \\
&\frac{16r(-112+(-2+r)r(294+r(-275+96r))))))))}{53760(-1+r)^2r^2})f_n \\
&+(\frac{157h^3+128h^3r+256h^3r^2-17408h^3r^3+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{50304h^3r^4-49920h^3r^5+16640h^3r^6}{53760(-1+r)^2r^2(-1+2r)})f_{n+r}+ \\
&\frac{8h^3(1-2r)^2(3+2r(2+r+3(-2+r)r^2))}{105(-1+2r)}f_{n+\frac{1}{2}}+ \\
&(\frac{-29h^3+128h^3r+256h^3r^2+512h^3r^3-3456h^3r^4+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{3840h^3r^5-1280h^3r^6}{53760(-1+r)^2r^2(-1+2r)})f_{n+1-r}+ \\
&(\frac{29h^3-128h^3r-350h^3r^2+2100h^3r^3+1680h^3r^4}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-672h^3r^5-93184h^3r^6+302080h^3r^7-390400h^3r^8}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{+227840h^3r^9-49152h^3r^{10}}{53760(-1+r)^2r^2(-1+2r)})f_{n+1}+ \\
&(\frac{-157h^4r+1535h^4r^2-4570h^4r^3-3640h^4r^4+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{56112h^4r^5-154112h^4r^6+215040h^4r^7-}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{168320h^4r^8+70400h^4r^9-12288h^4r^{10}}{53760(-1+r)^2r^2(-1+2r)})g_n+ \\
&(\frac{-29h^4r+255h^4r^2-730h^4r^3+840h^4r^4-}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{4816h^4r^5+28672h^4r^6-71680h^4r^7+87680h^4r^8}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-52480h^4r^9+12288h^4r^{10}}{53760(-1+r)^2r^2(-1+2r)})g_{n+1}+ \\
&\frac{-215040r^2+860160r^3-1075200r^4+430080r^5}{53760(-1+r)^2r^2(-1+2r)}z_n+ \\
&\frac{430080r^2-1720320r^3+2150400r^4-860160r^5}{53760(-1+r)^2r^2(-1+2r)}z_{n+\frac{1}{2}} \\
&+4z_{n+1},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
h^2\phi_{n+\frac{1}{2}} \quad &= \frac{h^3(47+2r^2(-925+2r(1615+392r(-5+2r))}{53760(-1+r)^2r^2(-1+2r)}f_n \\
&+\frac{-47h^3f_{n+r}+47h^3f_{n+1-r}}{53760(-1+r)^2r^2(-1+2r)}+ \\
&\frac{h^3(-47+2r^2(925-2r(1615+392r(-5+2r))}{53760(-1+r)^2r^2(-1+2r)}f_{n+1} \\
&+(\frac{47h^4r-365h^4r^2+990h^4r^3-1120h^4r^4+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{448h^4r^5}{53760(-1+r)^2r^2(-1+2r)})g_n+ \\
&(\frac{+47h^4r-365h^4r^2+990h^4r^3-1120h^4r^4+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{448h^4r^5}{53760(-1+r)^2r^2(-1+2r)})g_{n+1}+ \\
&\frac{-215040r^2+860160r^3-1075200r^4+430080r^5}{53760(-1+r)^2r^2(-1+2r)}z_n \\
&+(\frac{430080r^2-1720320r^3+2150400r^4}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-860160r^5}{53760(-1+r)^2r^2(-1+2r)})z_{n+\frac{1}{2}}+4z_{n+1},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
h^2\phi_{n+1-r} \quad &= (\frac{h^3(29+2r(-35+r(-245+8r(70+r(245+}{53760(-1+r)^2r^2} \\
&\frac{16r(28+(-2+r)r+154+r(-205+96r)))}{53760(-1+r)^2r^2})f_n \\
&+(\frac{29h^3-128h^3r-256h^3r^2-512h^3r^3+3}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{456h^3r^4-3840h^3r^5+1280h^3r^6}{53760(-1+r)^2r^2(-1+2r)})f_{n+r}+ \\
&\frac{8h^3(1-2r)^2(3+2r(2+r+3(-2+r)r^2))}{105(-1+2r)}f_{n+\frac{1}{2}} \\
&+(\frac{-157h^3-128h^3r-256h^3r^2+17408h^3r^3-}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{50304h^3r^4+49920h^3r^5-16640h^3r^6}{53760(-1+r)^2r^2(-1+2r)})f_{n+1-r} \\
&+(\frac{157h^3+128h^3r-14430h^3r^2+89140h^3r^3-}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{231280h^3r^4+243040h^3r^5+93184h^3r^6}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-517120h^3r^7+551680h^3r^8-263680h^3r^9}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{+49152h^3r^{10}}{53760(-1+r)^2r^2(-1+2r)})f_{n+1}+ \\
&(\frac{-29h^4r+255h^4r^2-730h^4r^3+840h^4r^4-4816h^4r^5+}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{28672h^4r^6-71680h^4r^7+87680h^4r^8-}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{52480h^4r^9+12288h^4r^{10}}{53760(-1+r)^2r^2(-1+2r)})g_n+ \\
&(\frac{-157h^4r+1535h^4r^2-4570h^4r^3-3640h^4r^4+56112h^4r^5}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-154112h^4r^6+215040h^4r^7-168320h^4r^8+70400h^4r^9}{53760(-1+r)^2r^2(-1+2r)} \\
&\frac{-12288h^4r^{10}}{53760(-1+r)^2r^2(-1+2r)})g_{n+1}+ \\
&\frac{-215040r^2+860160r^3-1075200r^4+430080r^5}{53760(-1+r)^2r^2(-1+2r)}z_n+ \\
&\frac{430080r^2-1720320r^3+2150400r^4-860160r^5}{53760(-1+r)^2r^2(-1+2r)}z_{n+\frac{1}{2}} \\
&+4z_{n+1},
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
h^2\phi_{n+1} \quad &= \frac{h^3(1-2r)^2(-29+70(-1+r)r(-1+8(-1+r)r))}{53760r^2(1-3r+2r^2)^2}f_n \\
&+\frac{(-29h^3+186h^3r)f_{n+r}}{53760r^2(1-3r+2r^2)^2}+ \\
&\frac{8h^3(-1+r)^2(3+14(-1+r)r)}{105(1-3r+2r^2)^2}f_{n+\frac{1}{2}}+ \\
&\frac{(157h^3-186h^3r)f_{n+1-r}}{53760r^2(1-3r+2r^2)^2}+ \\
&(\frac{-157h^3+186h^3r+14686h^3r^2-82160h^3r^3+}{53760r^2(1-3r+2r^2)^2} \\
&\frac{172120h^3r^4-157248h^3r^5+52416h^3r^6}{53760r^2(1-3r+2r^2)^2})f_{n+1}+ \\
&(\frac{29h^4r-313h^4r^2+1240h^4r^3-2300h^4r^4+2016h^4r^5}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-672h^4r^6}{53760r^2(1-3r+2r^2)^2})g_n+ \\
&(\frac{157h^4r-1849h^4r^2+7640h^4r^3}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-14460h^4r^4+12768h^4r^5-4256h^4r^6}{53760r^2(1-3r+2r^2)^2})g_{n+1}+ \\
&(\frac{215040r^2-1290240r^3+2795520r^4}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-2580480r^5+860160r^6}{53760r^2(1-3r+2r^2)^2})z_n \\
&+(\frac{-430080r^2+2580480r^3-5591040r^4+5160960r^5}{53760r^2(1-3r+2r^2)^2} \\
&\frac{-1720320r^6}{53760r^2(1-3r+2r^2)^2})z_{n+\frac{1}{2}}+4z_{n+1}.
\end{aligned} \tag{27}
$$

To maximize accuracy, the local truncation error of eq.(27) is minimized with respect to r. The optimal value of r is determined as

$$ r = \frac{1}{186}\left(93-\sqrt{2666}\right). \tag{28} $$

Substituting the optimal value of r into eqs. (16) to (27), we derive the final form of the implicit one-step optimized hybrid block method.

3. ANALYSIS OF THE METHOD

This section outlines the properties of the proposed method, which are essential for evaluating its effectiveness and reliability in solving general third-order IVPs.

3.1. ORDER AND ERROR CONSTANTS

The linear operator $\mathcal{L}$ associated with the proposed method, as defined in eqs. (16) to (27) is expressed as

$$
\begin{aligned}
\mathcal{L}[z(t_n);h] \quad &= \textstyle\sum_{j=0}[\mu_j z(t_n+jh)- \\
&h\delta_j z'(t_n+jh)-h^2\phi_j z''(t_n+jh)- \\
&h^3\zeta_j z'''(t_n+jh)-h^4\xi_j z''''(t_n+jh)],
\end{aligned} \tag{29}
$$

where $z(t_n)$ is an arbitrary function that is differentiable over $[a,b]$. Expanding eq. (29) as a Taylor series around t_n and grouping terms gives

$$
\begin{aligned}
\mathcal{L}[z(t_n);h] \quad &= C_0 z(t_n)+C_1 hz'(t_n)+\ldots+ \\
&C_p h^p z^p(t_n)+\ldots+C_{p+3}h^{p+3}z^{p+3}(t_n).
\end{aligned} \tag{30}
$$

According to Alkasassbeh *et al.* [5], the method in eqs. (16) to (27) is defined as having order p if the coefficients satisfy the following conditions
$C_0 = C_1 = C_2 = \ldots = C_{p+1} = C_{p+2} = 0$ and $C_{p+3} \neq 0$.
The vector C_{p+3} is referred to as the error constant, and $c_{p+3}h^{p+3}z^{p+3}(t_n)$ is the principal local truncation error at the point t_n. The method is associated with the following error constants

$$
\begin{aligned}
C_{10} = \big(&1.18614\times10^{-10}, 1.18614\times10^{-10}, \\
&7.40788\times10^{-10}, -1.43822\times10^{-11}, \\
&0, 1.43822\times10^{-11}, -7.40788\times10^{-10}, \\
&0, -7.39487\times10^{-9}, 7.52363\times10^{-9}, \\
&-7.39487\times10^{-9}, 0\big)^{\tau}.
\end{aligned}
$$

Hence, the method achieves an order of $p = 7$.

3.2. ZERO STABILITY

The stability of the method is analyzed by examining eqs. (16), (17), (18), and (23). The reformulated equation is given by

$$
\begin{aligned}
AZ_{\mu+1} \quad &= BZ_\mu + hG_0\Delta_\mu + h^2G_1\Phi_\mu + \\
&h^3[CF_{\mu+1}+DF_\mu+hEG_{\mu+1}+hHG_\mu],
\end{aligned} \tag{31}
$$

where A, B, C, D, E, G, and H are m × m matrices. The following vectors are defined as

$$
\begin{aligned}
Z_{\mu+1} &= [z_{n+r}, z_{n+\frac{1}{2}}, z_{n+1-r}, z_{n+1}], \\
Z_\mu &= [z_{n-r}, z_{n-\frac{1}{2}}, z_{n-1-r}, z_n], \\
F_{\mu+1} &= [f_{n+r}, f_{n+\frac{1}{2}}, f_{n+1-r}, f_{n+1}], \\
F_\mu &= [f_{n-r}, f_{n-\frac{1}{2}}, f_{n-1-r}, f_n], \\
G_{\mu+1} &= [g_{n+r}, g_{n+\frac{1}{2}}, g_{n+1-r}, g_{n+1}], \\
G_\mu &= [g_{n-r}, g_{n-\frac{1}{2}}, g_{n-1-r}, g_n], \\
\Delta_\mu &= [\delta_{n-r}, \delta_{n-\frac{1}{2}}, \delta_{n-1-r}, \delta_n], \\
\Phi_\mu &= [\phi_{n-r}, \phi_{n-\frac{1}{2}}, \phi_{n-1-r}, \phi_n].
\end{aligned}
$$

It is crucial to emphasize that zero stability refers to the stability behavior as $h \to 0$. In this context, the stability condition can be written as

$$ \rho(\lambda) = det[\lambda A - B]. \tag{32} $$

The first characteristic polynomial is given by

$$
\rho(\lambda) = \left| \lambda \begin{pmatrix} 1 & -\frac{193}{279} & 0 & -\frac{1}{558}(86-3\sqrt{2666}) \\ 0 & -\frac{193}{279} & 1 & -\frac{1}{558}(3\sqrt{2666}+86) \\ 0 & -4 & 0 & 1 \\ 0 & 8 & 0 & -4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{558}(3\sqrt{2666}+86) \\ 0 & 0 & 0 & -\frac{1}{558}(86-3\sqrt{2666}) \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -4 \end{pmatrix} \right|.
$$

This simplifies to

$$ \rho(\lambda) = -8\lambda^3(\lambda+1). \tag{33} $$

By solving eq. (33), we find that $\lambda = 0, 0, 0, -1$.

Definition 1. *Hybrid block method is considered to be zero-stable if the roots $\lambda_i, i = 1, 2, ..., s$ of the first characteristic polynomial $\rho(\lambda)$ satisfy $|\lambda_i| \leq 1$ and for those roots with $|\lambda_i| = 1$, their multiplicity must not exceed the order of the differential equation being solved (see Yakubu [11]).*

This shows that the method is zero-stable.

3.3. CONSISTENCY

Definition 2 (Lambert [12])**.** *A hybrid block method is considered to be consistent, if it has order $p \geq 1$.*

Thus, the one-step hybrid block method has order $p = 7 > 1$. This indicates that the method is consistent.

3.4. CONVERGENCE

Theorem 2 (Dahlquist [13])**.** *The necessary and sufficient conditions for numerical method to be convergent are that they must be zero-stable and consistent.*

Since the method meets both the conditions of zero-stability and consistency by the above theorem 2, we concluded that the method is convergent.

3.5. LINEAR STABILITY

The stability region of a numerical method provides insight into its behavior in the complex plane, particularly regarding the growth or decay of errors introduced during computation. This is determined by applying the method in eq. (31) to the test equations $z' = \lambda z$, $z'' = \lambda^2 z$, $z''' = \lambda^3 z$ and $z'''' = \lambda^4 z$, where $\lambda \in \mathcal{R}$. By setting $\varphi = \lambda h$, we obtain

$$Z_{\mu+1} = M(\varphi) Z_\mu,$$

where

$$\begin{aligned} M(\varphi) &= (A - \varphi^3 C - \varphi^4 E)^{-1}.(B + \varphi G_0 \\ &+\varphi^2 G_1 + \varphi^3 D + \varphi^4 H), \end{aligned}$$

is the amplification matrix. By analyzing the spectral radius, we determined the stability region, which is depicted in Figure 1

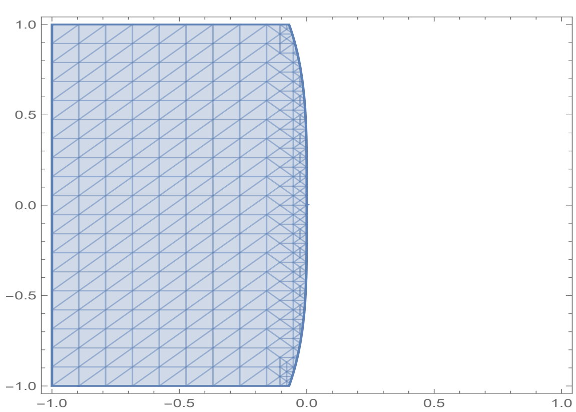


Figure 1. Stability region.

Definition 3 (Lambert [12])**.** *A numerical method is said to be A-stable if its region of absolute stability contains the entire negative complex half-plane.*

The stability region of the method lies entirely within the negative complex half-plane, confirming that the method is *A*-stable according to Ref. [12].

4. RESULTS AND DISCUSSION

This section evaluates the performance and efficiency of the new technique, with the primary objective of demonstrating the improved accuracy achieved by the proposed method.
The method is referred to as the Implicit One-Step Fourth-Derivative Hybrid Block Method (IOFDHBM).

Example 1.
Consider the linear stiff IVP [1]

$$\begin{aligned} &z''' + 5z'' + 7z' + 3z = 0, \quad z(0) = 1, \\ &z'(0) = 0, \quad z''(0) = -1. \end{aligned}$$

The exact solution is $z(t) = e^{-t} + te^{-t}$.

Example 2.
Consider the following linear stiff system, which has been solved in the literature by [3, 14]

$$\begin{aligned} z_1''' &= \tfrac{1}{68}(817z_1 + 1393z_2 + 448z_3), \\ & z_1(0) = 2,\ z_1'(0) = -12,\ z_1''(0) = 20, \\ z_2''' &= -\tfrac{1}{68}(1141z_1 + 2837z_2 + 896z_3), \\ & z_2(0) = -2,\ z_2'(0) = 28,\ z_2''(0) = -52, \\ z_3''' &= \tfrac{1}{136}(3059z_1 + 4319z_2 + 1592z_3), \\ & z_3(0) = -12,\ z_3'(0) = -33,\ z_3''(0) = 5. \end{aligned}$$

The exact solutions for this system are given as

$$\begin{aligned} z_1(t) &= e^t - 2e^{2t} + 3e^{-3t}, \\ z_2(t) &= 3e^t + 2e^{2t} - 7e^{-3t}, \\ z_3(t) &= -11e^t - 5e^{2t} + 4e^{-3t}. \end{aligned}$$

Example 3.
Consider the linear stiff IVP

$$z''' + z' = 0,\ z(0) = 0,\ z'(0) = 1,\ z''(0) = 2.$$

The exact solution is $z(t) = 2(1 - cos(t)) + sin(t)$.

Example 4.
Consider the non-linear IVP

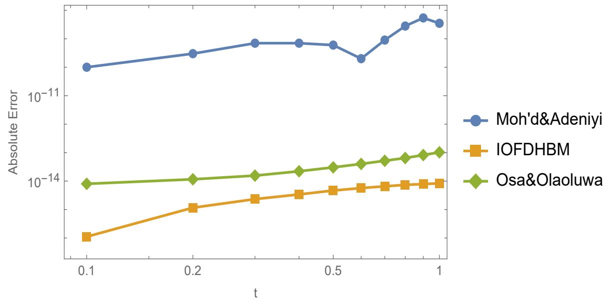
$$\begin{aligned} &z''' + zz'' - z' = 0, \\ &z(0) = 1,\ z'(0) = -1,\ z''(0) = 2. \end{aligned}$$

Example 5.
Application to the non-linear Genesio equation: The following nonlinear Genesio equation, initially introduced as a chaotic system by Genesio *et al.* [15] is considered by Allogmany *et al.* [3].

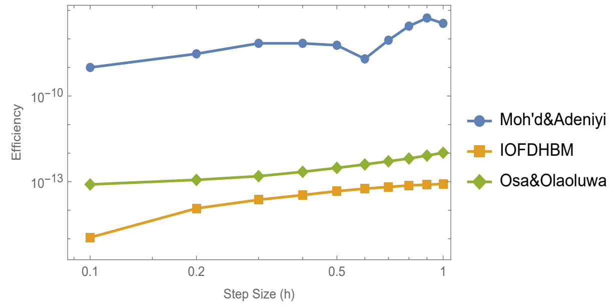
$$z''' = -\alpha z'' - \beta z' + f(z(t)),$$

where

$$f(z(t)) = -\gamma(t) + z(t)^2, \quad t \in [0, b],$$



(a) Absolute errors.



(b) Efficiency curves.

Figure 2. Comparison of absolute errors and efficiency for Example 1.

subject to

$$z(t_0) = 0.2, \quad z'(t_0) = -0.3, \quad z''(t_0) = 0.1,$$

with positive constants α, β, and γ satisfying $\alpha\beta \leq \gamma$.
Since the analytical solution to the Genesio equation is not available, we use the IOFDHBM to solve the equation with the following parameters: $t_0 = 0$, $\alpha = 1.2$, $\beta = 2.92$, $\gamma = 6$ using $h = 0.1$ over the interval $t \in [0, 1]$.

Example 6.

Application to the boundary layer in physics and fluid mechanics:
In physics and fluid mechanics, a boundary layer which refers to the fluid layer adjacent to a boundary surface where viscosity has a significant effect on the flow. The boundary layer equation, a third-order non-linear differential equation is given by

$$2z''' + zz'' = 0,$$

subject to

$$z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 1.$$

The Blasius equation, which describes boundary layer flow over a flat plate, has been extensively studied by researchers such as [3]. The objective is to solve this equation using the IOFDHBM with $h = 0.1$ over the time interval $t \in [0, 5]$.

Example 7.

Application to the thin film flow of a liquid on a surface:
Consider the engineering and physical problem of thin film flow of liquid on a surface, extensively studied by researchers such as [3, 14]. The fluid motion on a plane surface, moving in the same direction along the plane, is governed by third-order ODEs

$$z''' = f(z(t)), \tag{34}$$

where $z(t)$ represents the motion of the fluid, and the function $f(z(t))$ depends on the specific physical context, such as

$$f(z(t)) = z^{-2} - 1$$

for a fluid-draining problem on a dry surface, or

$$f(z(t)) = (1 + \epsilon + \epsilon^2)z^{-2} - (\epsilon + \epsilon^2)z^{-3} - 1$$

for a fluid-draining problem on a wet surface, where $\epsilon > 0$ represents the film thickness.

In problems related to thin film flow with a free surface of viscous fluid, the third-order ODE governing the free surface shape is given by

$$z''' = z^{-\rho}, \quad t \geq t_0, \tag{35}$$

subject to

$$z(t_0) = \mu_1, \quad z'(t_0) = \mu_2, \quad z''(t_0) = \mu_3,$$

where μ_1, μ_2, and μ_3 are constants. This problem, which has been solved by various authors, involves the conditions $z(t_0) = z'(t_0) = z''(t_0) = 1$, with $t_0 = 0$.
To solve this problem using the IOFDHBM, we consider $\mu = 2$.

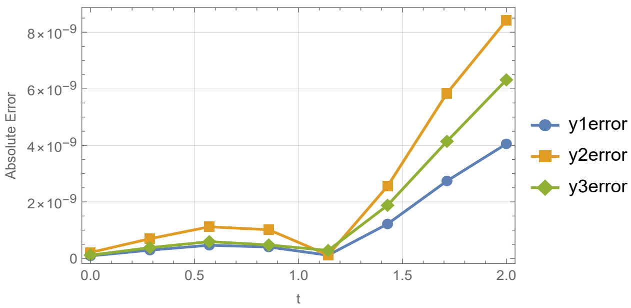
t	Error in [1]	Error in [16]	IOFDHBM
0.1	1.0000×10^{-10}	7.9936×10^{-15}	1.1100×10^{-16}
0.2	3.0000×10^{-10}	1.1546×10^{-14}	1.1569×10^{-15}
0.3	7.0000×10^{-10}	1.5543×10^{-14}	2.3221×10^{-15}
0.4	7.0000×10^{-10}	2.2093×10^{-14}	3.3964×10^{-15}
0.5	6.0000×10^{-10}	3.0309×10^{-14}	4.6532×10^{-15}
0.6	2.0000×10^{-10}	4.0079×10^{-14}	5.6758×10^{-15}
0.7	9.0000×10^{-10}	5.1625×10^{-14}	6.5464×10^{-15}
0.8	2.8000×10^{-9}	6.4615×10^{-14}	7.3446×10^{-15}
0.9	5.4000×10^{-9}	8.2045×10^{-14}	7.8574×10^{-15}
1	3.5000×10^{-9}	1.0258×10^{-13}	8.3166×10^{-15}

Table 1. Comparison of absolute errors for Example 1, $h = 0.1$.

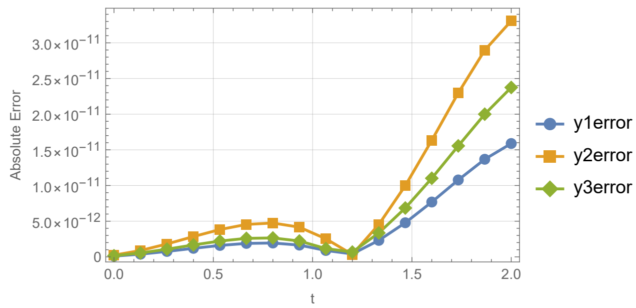
The results presented in Table 1 show that the IOFDHBM offers higher accuracy than the hybrid block methods developed by Mohammed *et al.* [1] (with order $p = 7$) and Osa *et al.* [16] (with order $p = 6$). Notably, the new method not only exhibits superior accuracy but also greater efficiency when compared to the aforementioned methods, as illustrated in Figures 2(a) and 2(b).

h	IOFDHBM	Error in [14]
$\frac{1}{4}$	6.3158×10^{-9}	5.7164×10^{-4}
$\frac{1}{8}$	3.3073×10^{-11}	4.8478×10^{-8}
$\frac{1}{16}$	9.9276×10^{-12}	4.1669×10^{-10}

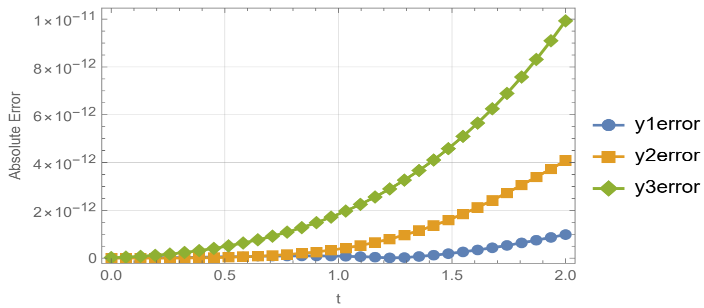
Table 2. Comparison of the maximum errors for Example 2 over the interval $t \in [0, 2]$.



(a) Absolute errors using $h = \frac{1}{4}$.

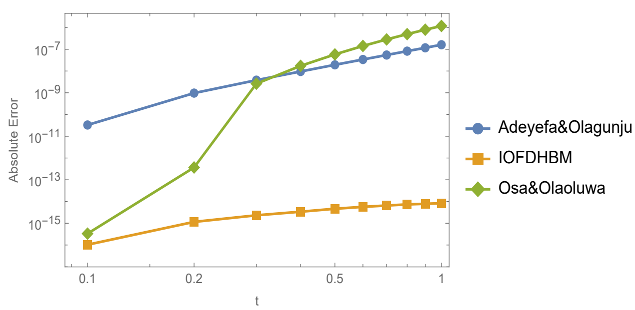


(b) Absolute errors using $h = \frac{1}{8}$.

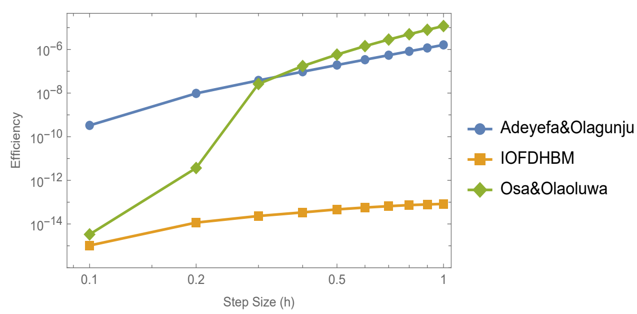


(c) Absolute errors using $h = \frac{1}{16}$.

Figure 3. Absolute errors for Example 2 at different step sizes.



(a) Absolute errors.



(b) Efficiency curves.

Figure 4. Comparison of absolute errors and efficiency for Example 3.

In Table 2, the IOFDHBM, formulated with three off-step points, demonstrates superior performance compared to the three-step hybrid method with order $p = 8$ developed by Jikantoro *et al.* [14]. Example 2 was solved over the interval $t \in [0, 2]$ using various step sizes to highlight the consistency of these techniques. The plot of absolute errors for the IOFDHBM in Example 2 is presented in Figure 3(a), 3(b), and 3(c), illustrating the stability of the solution at step sizes $h = 0.25,\ 0.125,\ 0.0625$ within the interval $t \in [0, 2]$.

Table 3 presents the results obtained for Example 3 over the interval [0, 1] with $h = 0.1$. The results show a significant difference, indicating that the IOFDHBM is more accurate than the methods discussed by Olagunju *et al.* [7] and Osa *et al.* [16]. Figure 4(a) displays the absolute error plot for Example 3, highlighting the performance of the IOFDHBM compared to the existing methods. Figure 4(b) presents the efficiency curve, further emphasizing the comparative performance of these methods.

Table 3. Comparison of absolute errors for Example 3.

t	Error in [7]	Error in [16]	IOFDHBM
0.1	3.3000×10^{-11}	3.3307×10^{-16}	4.5000×10^{-18}
0.2	9.6700×10^{-10}	3.6746×10^{-13}	8.6000×10^{-18}
0.3	3.8000×10^{-9}	2.6199×10^{-9}	9.0000×10^{-18}
0.4	9.5300×10^{-9}	1.7307×10^{-8}	5.7000×10^{-18}
0.5	1.9300×10^{-8}	5.8127×10^{-8}	1.4000×10^{-18}
0.6	3.3900×10^{-8}	1.4214×10^{-7}	1.2100×10^{-17}
0.7	5.4500×10^{-8}	2.8366×10^{-7}	2.6400×10^{-17}
0.8	8.2200×10^{-8}	4.9755×10^{-7}	4.4200×10^{-17}
0.9	1.1700×10^{-7}	7.9519×10^{-7}	6.5300×10^{-17}
1	1.6200×10^{-7}	1.1822×10^{-6}	8.9500×10^{-17}

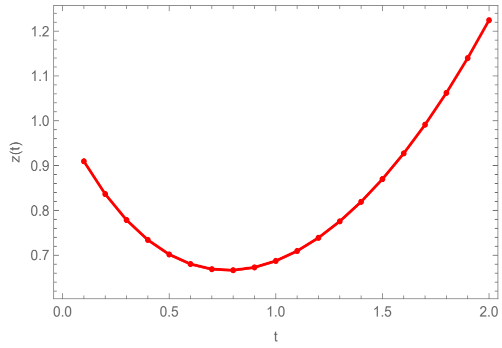


Figure 5. Approximation solution for Example 4.

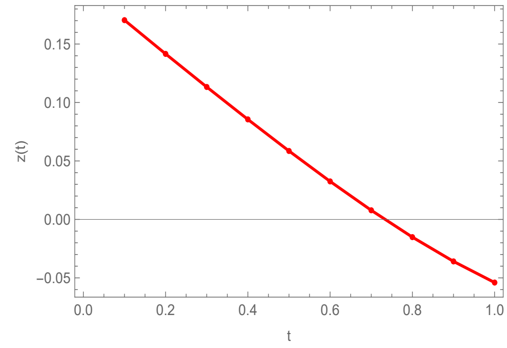


Figure 6. Approximation solution for Example 5.

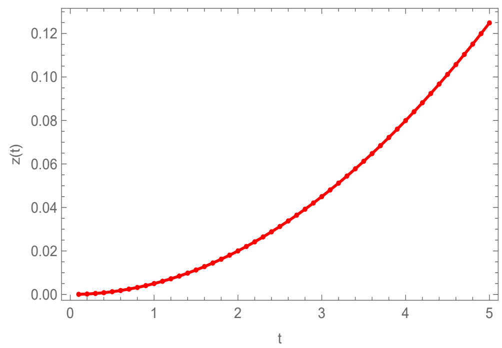
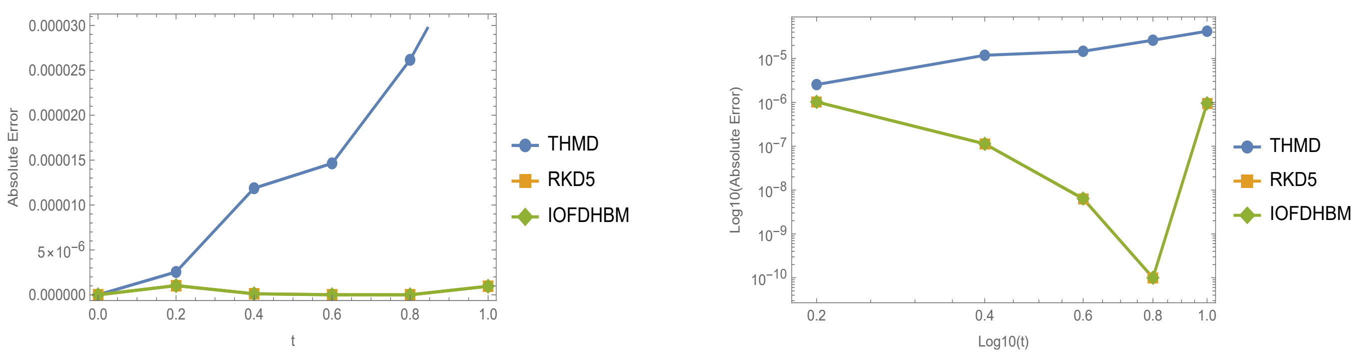


Figure 7. Approximation solution for Example 6.



(a) Absolute errors.

(b) Efficiency curves.

Figure 8. Comparison of absolute errors and efficiency for Example 7.

Table 4. Comparison of the approximation solutions for Example 7 using $h = 0.01$.

t	Exact Solution Meshee *et al.* [17]	RKD5 [17]	THMD [14]	IOFDHBM
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.221211030	1.22121100045	1.2212084858	1.2212100045
0.4	1.488834893	1.4888347799	1.4888467642	1.4888347798
0.6	1.807361404	1.8073613977	1.8073467642	1.8073613976
0.8	2.179819234	2.1798192339	2.1797930619	2.1798192339
1.0	2.608275822	2.6082748676	2.6082338883	2.6082748675

The plot in Figure 5 shows the numerical result for example 4, computed over the interval $[0, 2]$ with $h = 0.1$. The solution initially decreases towards $t = 1$ and then increases towards $t = 2$, highlighting the stiffness of Example 4. In this context, stiffness refers to the rapid changes in the solution relative to variations in the independent variable or parameters, which can present challenges for numerical computation.

Figure 6 shows the numerical solution for Example 5, computed over the interval $t \in [0, 1]$ with $h = 0.1$. The plot indicates that as the interval progresses from $t = 0$ to $t = 1$, the solution exhibits a slight decrease.

Figure 7 shows the numerical solution for solving Example 6,

computed over the interval $t \in [0, 5]$ with $h = 0.01$.
Table 4 presents the numerical results obtained from three different methods. The results shown in Figure 8(a) demonstrate that IOFDHBM achieves higher accuracy than THMD and RKD5. Additionally, Figure 8(b) illustrates the efficiency of all three methods in solving Example 7. A noticeable difference in accuracy is observed, with IOFDHBM outperforming THMD and RKD5, which have orders $p = 8$ and $p = 6$, respectively.

5. CONCLUSION

This study introduces an implicit single-step optimized fourth-derivative hybrid block method (IOFDHBM) for solving general third-order initial value problems (IVPs). The analysis confirms that the method is convergent, ensuring its robustness for a wide range of IVPs. The results presented in Tables 1 to 4, along with Figures 2, 4, and 8, demonstrate that the IOFDHBM consistently delivers highly accurate solutions while maintaining computational efficiency. Compared to existing methods, the IOFDHBM offers superior accuracy and stability due to the integration of optimization techniques in its formulation. In conclusion, the IOFDHBM is a reliable and effective numerical technique for solving third-order IVPs, providing a significant improvement over previously developed methods.

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