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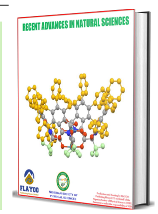

A novel one-step optimized hybrid block method for solving general second-order ordinary differential equations

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ABSTRACT

This study introduces a novel one-step hybrid block method for solving both stiff and non-stiff second-order ordinary differential equations (ODEs). The method incorporates optimization techniques and higher derivative functions, addressing the computational challenges of stiff ODEs while ensuring key numerical properties such as zero-stability, A-stability, and convergence. Numerical experiments demonstrate the method's effectiveness across various problems, including classical examples from heat conduction, electrical circuits, and the Van der Pol oscillator. The results reveal that the proposed method achieves superior accuracy and efficiency, significantly outperforming existing methods in the literature. These findings underscore the potential of this approach as a robust and versatile tool for solving a wide variety of practical ODEs in engineering and applied sciences.

1. INTRODUCTION

Many phenomena in science, medicine, and engineering such as spring dynamics, electrical circuits, and projectile motion, are modeled by second-order ordinary differential equations (ODE)s. These ODEs are often too complex for analytical solutions, making advanced numerical methods such as block methods essential. Since numerical methods are inherently approximations, their careful derivation is crucial for minimizing errors.

Traditionally, higher-order ODEs are solved by converting them into systems of first-order IVPs. However, this approach introduces challenges such as increased computational effort and inefficiency, as noted by Lambert [1] and Awoyemi [2]. Manual implementation of schemes in traditional methods can also be time-consuming. In contrast, numerical schemes offer a faster and more efficient means of approximating solutions. To overcome these limitations, advanced numerical methods have been developed. For instance, Allogmany *et al.* [3] introduced a two-step block method that includes third- and fourth-derivative terms for solving general second-order ODEs. In a similar vein, hybrid block methods, which address the limitations of traditional linear multistep methods in handling higher-order ODEs, have gained prominence. Rufai and Ramos [4] introduced a one-step hybrid block method that integrates a third-derivative term and employs three equally spaced off-step points for collocation. This method has demonstrated high efficiency in solving complex problems, including Bratu's and Troesch's equations.

Olabode and Momoh [5] developed a two-step Chebyshev hybrid multistep method for directly solving second-order IVPs and boundary value problems (BVPs). This method utilized

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four equally spaced off-step points and employed the Chebyshev polynomial of the first kind as the basis function in its derivation. Abolarin *et al*. [6] introduced a two-step implicit hybrid block method with four off-step points, designed for directly solving second-, third-, and fourth-order ODEs. Similarly, Omar and Abdelrahim [7] introduced a single-step hybrid block method with generalized three off-step points for solving second-order ODEs.

Qureshi *et al*. [8] introduced a one-step optimized block method with a fifth algebraic order of convergence for solving first- and higher-order IVPs numerically. This method was implemented in both fixed and variable step size modes to achieve improved accuracy. Emmanuel and Qureshi [9] recently proposed a numerical integration method utilizing an interpolating function involving a transcendental function of exponential type to solve continuous dynamical systems of ODEs. In another study, Qureshi *et al*. [10] developed a single-step nonlinear numerical method tailored for first-order singular and stiff differential problems, which was implemented in adaptive mode. Additionally, Abuasbeh *et al*. [11] designed an optimized family of one-step block techniques with two optimal points for solving first-order models of infectious diseases. These techniques, implemented in both fixed and adaptive step size modes, demonstrated superior performance compared to other methods.

Ramos and Singh [12] presented a two-step optimized third-derivative block hybrid method incorporating two off-grid points, specifically intended for solving general second-order BVPs. Singla *et al*. [13] developed an optimized two-step hybrid block method that was implemented in a variable step-size mode for second-order ODEs. Singh and Ramos [14] introduced a two-step block hybrid method that utilized two optimal points for solving IVPs. More recently, Akinnukawe and Okunuga [15] proposed a single-step block hybrid method with two optimal points for solving second-order ODEs.

Most methods in the literature lack both high-order accuracy and stability. These limitations have driven ongoing efforts to improve the accuracy and efficiency of numerical techniques for directly solving higher-order IVPs. Building on this progress, the present study aims to develop a single-step optimized hybrid block method that leverages four carefully selected optimal points to enhance both stability and accuracy in solving general second-order IVPs.

The primary objective is to overcome the shortcomings of existing approaches and provide a more efficient and reliable solution for directly addressing second-order IVPs. The robustness of the proposed method is demonstrated through a comprehensive analysis of key properties, including convergence. Additionally, numerical experiments are conducted to evaluate its performance, highlighting its superior accuracy and effectiveness compared to existing methods.

The structure of this study is as follows: Section 2 presents the derivation of the proposed method; Section 3 examines the method's key properties; Section 4 presents the implementation strategy; Section 5 discusses the numerical results from applying the method to various test problems, and Section 6 concludes the study by summarizing the findings and highlighting the method's advantages over existing approaches.

2. DEVELOPMENT OF THE METHOD

This section outlines the derivation of a single-step optimized block hybrid method, utilizing four optimal points, for solving general second-order IVPs of the form

$$z''(t) = f(t, z, z'), \quad z(t_0) = z_0, \quad z'(t_0) = z'_0, \\ t \in [a, b], \tag{1}$$

where $f \in \mathbb{R}$ is a sufficiently differentiable function that satisfies the Lipschitz condition.

lemma 1 (Lipschitz condition). *Suppose the second-order ODE is given in the form of Equation (1). The Lipschitz condition on f with respect to z and z' is as follows:*

$$|f(t, z_1, \eta_1) - f(t, z_2, \eta_2)| \leq \mathbf{L}(|z_1 - z_2| + |\eta_1 - \eta_2|), \tag{2}$$

for all $(t, z_1, \eta_1), (t, z_2, \eta_2)$ in some domain $\mathbf{D}$, where $L > 0$ is a constant.

Theorem 1 (Existence and Uniqueness of Solution). *Suppose the second-order ODE is given in the form of Equation (1). If the following conditions hold:*

1. *The function $\mathbf{f}(t, z, \eta)$ is continuous in t, z, η within a domain $\mathbf{D} \subset \mathbb{R}^3$ containing (t_0, z_0, η_0).*
2. *The function $\mathbf{f}(t, z, \eta)$ satisfies a Lipschitz condition in z & η, as given in Equation (2), for all $(t, z_1, \eta_1), (t, z_2, \eta_2) \in \mathbf{D}$.*

Then, there exists a unique solution $z(t)$ to the ODE in some interval $[t_0 - \delta, t_0 + \delta]$, where $\delta > 0$ depends on $\mathbf{f}$ and the initial conditions.

Proof
Step 1: Reduction to a first-order system.
Introduce a new variable $\eta = z'$. The second-order ODE becomes

$$z' = \eta, \\ \eta' = \mathbf{f}(t, z, \eta). \tag{3}$$

This is now a system of first-order ODEs:

$$\mathbf{Z}' = \mathbf{F}(t, \mathbf{Z}), \tag{4}$$

where:

$$\mathbf{Z} = \begin{bmatrix} z \\ \eta \end{bmatrix}, \ \mathbf{F}(t, \mathbf{Z}) = \begin{bmatrix} \eta \\ \mathbf{f}(t, z, \eta) \end{bmatrix}.$$

Step 2: Continuity and Lipschitz condition.
• If $\mathbf{f}(t, z, \eta)$ is continuous, then $\mathbf{F}(t, \mathbf{Z})$ is continuous.
• If $\mathbf{f}(t, z, \eta)$ satisfies the Lipschitz condition in z and η, then $\mathbf{F}(t, \mathbf{Z})$ satisfies the Lipschitz condition.

$$\|\mathbf{F}(t, \mathbf{Z_1}) - \mathbf{F}(t, \mathbf{Z_2})\| \leq \mathbf{L}\|\mathbf{Z_1} - \mathbf{Z_2}\|, \tag{5}$$

where $\|.\|$ denotes the Euclidean norm.
Step 3: Application of the Picard-Lindelöf theorem [16].
The Picard-Lindelöf theorem states that if $\mathbf{F}(t, \mathbf{Z})$ is continuous and satisfies a Lipschitz condition in $\mathbf{Z}$, then the system:

$$\mathbf{Z}' = \mathbf{F}(t, \mathbf{Z}), \quad \mathbf{Z}(t_0) = \begin{bmatrix} z_0 \\ \eta_0 \end{bmatrix}, \tag{6}$$

has a unique solution $\mathbf{Z}(t)$ in some interval $[t_0-\delta, t_0+\delta]$, where $\delta > 0$.

Step 4: Reconstructing the solution to the original ODE.
The unique solution $\mathbf{Z}(t)$ is given by:

$$\mathbf{Z}(t) = \begin{bmatrix} z(t) \\ \eta(t) \end{bmatrix}, \tag{7}$$

where $z(t)$ satisfies the original second-order ODE and the initial conditions

$$z(t_0) = z_0, \; z'(t_0) = \eta_0.$$

Thus $z(t)$ is the unique solution to the original second-order ODE.

To develop the method, we approximate the solution using a power series polynomial $Z(t)$ expressed as:

$$z(t) \approx Z(t) = \sum_{j=0}^{9} a_j t^j, \tag{8}$$

where a_j are the unknown coefficients, which are determined by applying interpolation and collocation conditions at the specific points. The solution is approximated over the interval $a \le t \le b$, where

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

The second and third derivatives of $Z(t)$ are expressed as

$$z''(t) = \sum_{j=2}^{9} j(j-1)a_j t^{j-2}, \tag{9}$$

and

$$z'''(t) = \sum_{j=3}^{9} j(j-1)(j-2)a_j t^{j-3}. \tag{10}$$

To determine the coefficients a_j, equation (8) is interpolated at $t = t_{n+j}, j = 0, 1$, while equation (9) is collocated at $t = t_{n+j}, j = 0, r, s, 1-s, 1-r, 1$. Similarly, equation (10) is collocated at $t = t_{n+j}, j = 0, 1$. This setup leads to a system of ten equations with ten unknowns, from which the coefficients a_j are computed. The resulting expressions for the interpolated and collocated approximations are given as follows:

$$Z(t) = z(t_{n+j}), j = 0, 1, \tag{11}$$

$$Z''(t) = f(t_{n+j}), j = 0, r, s, 1-s, 1-r, 1, \tag{12}$$

$$Z'''(t) = \eta(t_{n+j}), j = 0, 1. \tag{13}$$

This system of ten equations can be written in matrix form as

$$AB = F, \tag{14}$$

where

$$A = \begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 & t_n^8 & t_n^9 \\ 1 & t_{n+1} & t_{n+1}^2 & t_{n+1}^3 & t_{n+1}^4 & t_{n+1}^5 & t_{n+1}^6 & t_{n+1}^7 & t_{n+1}^8 & t_{n+1}^9 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 & 72t_n^7 \\ 0 & 0 & 2 & 6t_{n+r} & 12t_{n+r}^2 & 20t_{n+r}^3 & 30t_{n+r}^4 & 42t_{n+r}^5 & 56t_{n+r}^6 & 72t_{n+r}^7 \\ 0 & 0 & 2 & 6t_{n+s} & 12t_{n+s}^2 & 20t_{n+s}^3 & 30t_{n+s}^4 & 42t_{n+s}^5 & 56t_{n+s}^6 & 72t_{n+s}^7 \\ 0 & 0 & 2 & 6t_{n+1-s} & 12t_{n+1-s}^2 & 20t_{n+1-s}^3 & 30t_{n+1-s}^4 & 42t_{n+1-s}^5 & 56t_{n+1-s}^6 & 72t_{n+1-s}^7 \\ 0 & 0 & 2 & 6t_{n+1-r} & 12t_{n+1-r}^2 & 20t_{n+1-r}^3 & 30t_{n+1-r}^4 & 42t_{n+1-r}^5 & 56t_{n+1-r}^6 & 72t_{n+1-r}^7 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 & 56t_{n+1}^6 & 72t_{n+1}^7 \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 & 336t_n^5 & 504t_n^6 \\ 0 & 0 & 0 & 6 & 24t_{n+1} & 60t_{n+1}^2 & 120t_{n+1}^3 & 210t_{n+1}^4 & 336t_{n+1}^5 & 504t_{n+1}^6 \end{pmatrix},$$

$$B = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]^T,$$

and

$$F = [z_n, z_{n+1}, f_n, f_{n+r}, f_{n+s}, f_{n+1-s}, f_{n+1-r}, f_{n+1}, \eta_n, \eta_{n+1}]^T.$$

Equations (11)-(13) are solved to determine the coefficients a_j, which are then substituted into equation (8). After performing algebraic simplifications, the continuous approximation equation is obtained as:

$$Z(t) = \sum_{j=0,1} \Phi_j(t) z_{n+j} + h^2 \sum_{j=0,r,s,1-s,1-r,1} \Psi_j(t) f_{n+j} + h^3 \sum_{j=0,1} \Upsilon_j(t)\eta_{n+j}, \tag{15}$$

where n is the grid index, $n = 0, 1, 2, \ldots, N-1$, and h is defined as $h = t_{n+1} - t_n$ within the interval $[a, b]$. Equation (15) alone does not suffice for solving general second-order IVPs, an additional equation is derived by differentiating equation (15) with respect to t.

$$z'(t) = \frac{1}{h}\left[\sum_{j=0,1} \Phi_j'(t) z_{n+j} + h^2 \sum_{j=0,r,s,1-s,1-r,1} \Psi_j'(t) f_{n+j} + h^3 \sum_{j=0,1} \Upsilon_j'(t)\eta_{n+j} \right]. \tag{16}$$

We imposed the condition that

$$Z'(t) = \xi(t). \tag{17}$$

By evaluating equation (15) at $t_{n+j}, j = r, s, 1-s, 1-r$, and equation (16) at all collocation points, the general continuous schemes are obtained. However, due to the cumbersome nature of the expressions, the full continuous scheme is not presented here. To achieve maximum accuracy, we optimize the local truncation error $h\xi_{n+1}$ derived from the continuous scheme. This optimization leads to the expression

$$\mathcal{L}[z(t_{n+1});h] = \frac{(2+9(-1+s)s+}{101606400} \frac{r(-9-42(-1+s)s)+r^2(9+}{101606400} \frac{42(-1+s)s))h^{10}z^{(10)}[t_n]}{101606400} + \frac{(100+451(-1+s)s-11r(41+}{10059033600} \frac{192(-1+s)s)+11r^2(41+192(-1+s)}{10059033600} \frac{s))h^{11}z^{(11)}[t_n]}{10059033600} + O(h^{12}). \tag{18}$$

To determine the values of r and s, the coefficients of h^{10} and h^{11} in equation (18) are set to zero, and the resulting equations is solved simultaneously. The optimal values of r and s are

$$r = \frac{1}{66}\left(33 - \sqrt{33\left(4\sqrt{3}+9\right)}\right),$$

$$s = \frac{1}{66}\left(33 - \sqrt{33\left(9-4\sqrt{3}\right)}\right).$$

The solution satisfies the constraints $0 < r < s < 1-s < 1-r < 1$. The optimal values of r and s are substituted into equation (18), and this simplifies to:

$$\mathcal{L}[z(t_{n+1});h] = \frac{z^{(13)}(t_n)h^{13}}{237342897792000} + O(h^{14}).$$

Using this result, the equations for the hybrid block method are derived as follows:

$$\begin{aligned} z_{n+r} = &-\frac{h^2}{3659040\sqrt{69-16\sqrt{3}}}\times \\ &\Big[4\left(536\sqrt{3}+3\sqrt{66\left(35526-9991\sqrt{3}\right)}+2251\right)f_n \\ &+4\left(-536\sqrt{3}+3\sqrt{66(35526-9991\sqrt{3})}-2251\right)f_{n+1} \\ &+\left(39921-40858\sqrt{3}\right)\sqrt{4\sqrt{3}+9} \\ &+3\left(42382\sqrt{3}+29541\right)\sqrt{9-4\sqrt{3}}f_{n+1-s} \\ &+6776\sqrt{3(6021-3424\sqrt{3})}f_{n+1-r} \\ &+242\sqrt{9-4\sqrt{3}}\left(958\sqrt{3}-243\right)f_{n+r} \\ &+\left(40858\sqrt{3}-39921\right)\sqrt{4\sqrt{3}+9} \\ &+3\left(42382\sqrt{3}+29541\right)\sqrt{9-4\sqrt{3}}f_{n+s} \\ &-2\left(79\sqrt{3}+3\sqrt{176\sqrt{3}+759}-166\right)h\eta_n \\ &+2\left(-79\sqrt{3}+3\sqrt{176\sqrt{3}+759}+166\right)h\eta_{n+1}\Big] \\ &+\frac{1}{66}\left(\sqrt{33(4\sqrt{3}+9)}+33\right)z_n \\ &+\frac{1}{66}\left(33-\sqrt{33(4\sqrt{3}+9)}\right)\times z_{n+1}. \end{aligned} \tag{19}$$

$$\begin{aligned} z_{n+s} = &\frac{h^2}{3659040\sqrt{16\sqrt{3}+69}}\times \\ &\Big[-4\left(536\sqrt{3}+3\sqrt{66(9991\sqrt{3}+35526)}-2251\right)f_n \\ &-4\left(-536\sqrt{3}+3\sqrt{66(9991\sqrt{3}+35526)}+2251\right)f_{n+1} \\ &-6776\sqrt{3(3424\sqrt{3}+6021)}f_{n+1-s} \\ &+3\left(29541-42382\sqrt{3}\right)\sqrt{4\sqrt{3}+9} \\ &+\sqrt{3(984133664\sqrt{3}+6756706455)}f_{n+1-r} \\ &-3\sqrt{33(76041512\sqrt{3}+797096733)} \\ &+\sqrt{3(984133664\sqrt{3}+6756706455)}f_{n+r} \\ &-242\sqrt{4\sqrt{3}+9}\left(958\sqrt{3}+243\right)f_{n+s} \\ &+2h^3(\left(79\sqrt{3}-3\sqrt{759-176\sqrt{3}}+166\right)\eta_n \\ &+\left(79\sqrt{3}+3\sqrt{759-176\sqrt{3}}+166\right)\eta_{n+1})\Big] \\ &+\frac{1}{66}\left(\sqrt{33(9-4\sqrt{3})}+33\right)z_n \\ &-\frac{1}{66}\left(\sqrt{33(9-4\sqrt{3})}-33\right)z_{n+1}. \end{aligned} \tag{20}$$

$$\begin{aligned} z_{n+1-s} = &-\frac{h^2}{3659040\sqrt{16\sqrt{3}+69}}\times \\ &\Big[4\left(-536\sqrt{3}+3\sqrt{66(9991\sqrt{3}+35526)}+2251\right)f_n \\ &+4\left(536\sqrt{3}+3\sqrt{66(9991\sqrt{3}+35526)}-2251\right)f_{n+1} \\ &+\sqrt{4\sqrt{3}+9}\left(242\left(958\sqrt{3}+243\right)\right)f_{n+1-s} \\ &+\left(3\left(42382\sqrt{3}-29541\right)\right)\sqrt{4\sqrt{3}+9} \\ &+\left(40858\sqrt{3}+39921\right)\sqrt{9-4\sqrt{3}}f_{n+1-r} \\ &-\left(3\left(42382\sqrt{3}-29541\right)\right)\sqrt{4\sqrt{3}+9} \\ &+\left(40858\sqrt{3}+39921\right)\sqrt{9-4\sqrt{3}}f_{n+r} \\ &+\sqrt{4\sqrt{3}+9}\left(6776\left(17\sqrt{3}+18\right)\right)f_{n+s} \\ &+2h(\left(79\sqrt{3}+3\sqrt{759-176\sqrt{3}}+166\right)\eta_n \end{aligned}$$

$$
\begin{aligned}
&+\left(79\sqrt{3}-3\sqrt{759-176\sqrt{3}}+166\right)\eta_{n+1})\Big]\\
&+\frac{1}{66}\left(33-\sqrt{33(9-4\sqrt{3})}\right)z_n\\
&+\frac{1}{66}\left(\sqrt{33(9-4\sqrt{3})}+33\right)z_{n+1}. \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
z_{n+1-r} = &-\frac{h^2}{3659040\sqrt{69-16\sqrt{3}}}\\
&\Big[4\left(-536\sqrt{3}+3\sqrt{66(35526-9991\sqrt{3})}-2251\right)f_n\\
&+4\left(536\sqrt{3}+3\sqrt{66(35526-9991\sqrt{3})}+2251\right)f_{n+1}\\
&+\left(40858\sqrt{3}-39921\right)\sqrt{4\sqrt{3}+9}\\
&+3\left(42382\sqrt{3}+29541\right)\sqrt{9-4\sqrt{3}}f_{n+1-s}\\
&+242\sqrt{9-4\sqrt{3}}\left(958\sqrt{3}-243\right)f_{n+1-r}\\
&+6776\sqrt{3(6021-3424\sqrt{3})}f_{n+r}\\
&+\left(39921-40858\sqrt{3}\right)\sqrt{4\sqrt{3}+9}\\
&+3\left(42382\sqrt{3}+29541\right)\sqrt{9-4\sqrt{3}}f_{n+s}\\
&+2h\Big(\left(79\sqrt{3}-3\sqrt{176\sqrt{3}+759}-166\right)\eta_n\\
&+\left(79\sqrt{3}+3\sqrt{176\sqrt{3}+759}-166\right)\eta_{n+1}\Big)\Big]\\
&+\frac{1}{66}\left(33-\sqrt{33(4\sqrt{3}+9)}\right)z_n\\
&+\frac{1}{66}\left(\sqrt{33(4\sqrt{3}+9)}+33\right)z_{n+1}. \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
h\xi_n = &\frac{h^2}{3360}\left[-212f_n-4f_{n+1}\right.\\
&+(-366-27\sqrt{3}+\sqrt{3(9981-3704\sqrt{3})})f_{n+1-s}\\
&+(-366+27\sqrt{3}+\sqrt{3(3704\sqrt{3}+9981)})f_{n+1-r}\\
&-(366-27\sqrt{3}+\sqrt{3(3704\sqrt{3}+9981)})f_{n+r}\\
&-(366+27\sqrt{3}+\sqrt{3(9981-3704\sqrt{3})})f_{n+s}\\
&\left.-4h\eta_n\right]-z_n+z_{n+1}. \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
h\xi_{n+r} = &\frac{h^2}{13416480}\times\\
&[\ 12\left(3640\sqrt{3}+\sqrt{5092208\sqrt{3}+135720717}+4384\right)f_n\\
&+12\left(\sqrt{5092208\sqrt{3}+135720717}-8\left(455\sqrt{3}+548\right)\right)f_{n+1}\\
&-\left(106830-137089\sqrt{3}\right)\sqrt{4\sqrt{3}+9}\\
&+3\left(119747\sqrt{3}+181912\right)\sqrt{9-4\sqrt{3}}f_{n+1-s}\\
&+242\sqrt{9-4\sqrt{3}}\left(80\sqrt{3}-1437\right)f_{n+1-r}\\
&-242\sqrt{66\left(77513\sqrt{3}+198750\right)}f_{n+r}\\
&-\sqrt{7557400992\sqrt{3}+258650247087}\\
&+3\left(119747\sqrt{3}+181912\right)\sqrt{9-4\sqrt{3}}f_{n+s}\\
&+2h(\left(1120\sqrt{3}+\sqrt{16226397-871552\sqrt{3}}+837\right)\eta_n\\
&+\left(1120\sqrt{3}-\sqrt{16226397-871552\sqrt{3}}+837\right)\eta_{n+1})]\\
&-z_n+z_{n+1}. \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
h\xi_{n+s} = &-\frac{h^2}{13416480}\times\\
&\Big[12\left(3640\sqrt{3}+\sqrt{135720717-5092208\sqrt{3}}-4384\right)f_n\\
&+12\left(-3640\sqrt{3}+\sqrt{135720717-5092208\sqrt{3}}+4384\right)f_{n+1}\\
&+\sqrt{4\sqrt{3}+9}\left(242\left(80\sqrt{3}+1437\right)\right)f_{n+1-s}\\
&+\left(545736-359241\sqrt{3}\right)\sqrt{4\sqrt{3}+9}\\
&+\left(137089\sqrt{3}+106830\right)\sqrt{9-4\sqrt{3}}f_{n+1-r}\\
&+(\left(545736-359241\sqrt{3}\right)\\
&-\sqrt{9-4\sqrt{3}}\left(137089\sqrt{3}+106830\right))f_{n+r}\\
&+\sqrt{4\sqrt{3}+9}\cdot 242\left(1887-791\sqrt{3}\right)f_{n+s}\\
&+2h\Big(\left(1120\sqrt{3}+\sqrt{871552\sqrt{3}+16226397}-837\right)\eta_n\\
&+\left(1120\sqrt{3}-\sqrt{871552\sqrt{3}+16226397}-837\right)\eta_{n+1}\Big)\Big]\\
&-z_n+z_{n+1}. \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
h\xi_{n+1-s} = &\frac{h^2}{13416480}\times\\
&\Big[12\left(-3640\sqrt{3}+\sqrt{135720717-5092208\sqrt{3}}+4384\right)f_n\\
&+12\left(3640\sqrt{3}+\sqrt{135720717-5092208\sqrt{3}}-4384\right)f_{n+1}
\end{aligned}
$$

$$+ 242\sqrt{66\left(198750 - 77513\sqrt{3}\right)}f_{n+1-s} - \left(3\left(119747\sqrt{3} - 181912\right)\sqrt{4\sqrt{3}+9} + \sqrt{258650247087 - 7557400992\sqrt{3}}\right)f_{n+1-r} + \left(3\left(181912 - 119747\sqrt{3}\right)\sqrt{4\sqrt{3}+9} + \sqrt{258650247087 - 7557400992\sqrt{3}}\right)f_{n+r} + 242\sqrt{4\sqrt{3}+9}\left(80\sqrt{3}+1437\right)f_{n+s} + 2\left(-1120\sqrt{3} + \sqrt{871552\sqrt{3} + 16226397} + 837\right)h\eta_n - 2\left(1120\sqrt{3} + \sqrt{871552\sqrt{3} + 16226397} - 837\right)h\eta_{n+1}\Big] - z_n + z_{n+1}. \quad (26)$$

$$h\xi_{n+1-r} = \frac{h^2}{13416480}\times \Big[12\left(3640\sqrt{3} - \sqrt{5092208\sqrt{3} + 135720717} + 4384\right)f_n - 12\left(3640\sqrt{3} + \sqrt{5092208\sqrt{3} + 135720717} + 4384\right)f_{n+1} + \left(\sqrt{7557400992\sqrt{3} + 258650247087} + 3\left(119747\sqrt{3} + 181912\right)\sqrt{9 - 4\sqrt{3}}\right)f_{n+1-s} + 242\sqrt{66\left(77513\sqrt{3} + 198750\right)}f_{n+1-r} + 242\sqrt{33\left(652017 - 315332\sqrt{3}\right)}f_{n+r} + \left(\left(106830 - 137089\sqrt{3}\right)\sqrt{4\sqrt{3}+9} + 3\left(119747\sqrt{3} + 181912\right)\sqrt{9 - 4\sqrt{3}}\right)f_{n+s} + 2h\left(\left(1120\sqrt{3} - \sqrt{16226397 - 871552\sqrt{3}} + 837\right)\eta_n + \left(1120\sqrt{3} + \sqrt{16226397 - 871552\sqrt{3}} + 837\right)\eta_{n+1}\right)\Big]\times - z_n + z_{n+1}. \quad (27)$$

$$h\xi_{n+1} = \frac{h^2}{3360}\Big[4f_n + 212f_{n+1} + \left(27\sqrt{3} + \sqrt{3\left(9981 - 3704\sqrt{3}\right)} + 366\right)f_{n+1-s} + \left(-27\sqrt{3} + \sqrt{3\left(3704\sqrt{3} + 9981\right)} + 366\right)f_{n+1-r} - \left(27\sqrt{3} + \sqrt{3\left(3704\sqrt{3} + 9981\right)} - 366\right)f_{n+r} + \left(27\sqrt{3} - \sqrt{3\left(9981 - 3704\sqrt{3}\right)} + 366\right)f_{n+s} - 4h\eta_{n+1}\Big] - z_n + z_{n+1}. \quad (28)$$

3. ANALYSIS OF THE METHOD

This section presents the key properties of the method, including the truncation error, zero-stability, consistency, convergence, and linear stability characteristics. To facilitate this analysis, we reformulate the method into the following matrix equation

$$\Gamma_1 Z_{n+1} = \Gamma_0 Z_n + hN_0X_n + h^2(K_0F_n + K_1F_{n+1}) + h^3(P_0H_n + P_1H_{n+1}), \quad (29)$$

where Γ_0, Γ_1, K_0, K_1, P_0, P_1, and N_0 are 5×5 matrices. The coefficients of these matrices are defined as follows

$$\Gamma_0 = \begin{pmatrix} 0 & 0 & \cdots & \phi_{1,0} \\ 0 & 0 & \cdots & \phi_{2,0} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \phi_{5,0} \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,5} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,5} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{5,1} & \phi_{5,2} & \cdots & \phi_{5,5} \end{pmatrix}.$$

$$K_1 = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,5} \\ \psi_{2,1} & \psi_{2,2} & \cdots & \psi_{2,5} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{5,1} & \psi_{5,2} & \cdots & \psi_{5,5} \end{pmatrix},$$

$$K_0 = \begin{pmatrix} 0 & 0 & \cdots & \psi_{1,0} \\ 0 & 0 & \cdots & \psi_{2,0} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_{5,0} \end{pmatrix}.$$

$$N_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{pmatrix},$$

$$P_0 = \begin{pmatrix} 0 & 0 & \cdots & \upsilon_{1,0} \\ 0 & 0 & \cdots & \upsilon_{2,0} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \upsilon_{5,0} \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 0 & \cdots & \upsilon_{1,1} \\ 0 & 0 & \cdots & \upsilon_{2,1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \upsilon_{5,1} \end{pmatrix}.$$

The vectors Z_{n+1}, Z_n, F_n, F_{n+1}, H_n, H_{n+1}, and X_n are defined as follows:

$$Z_n = (z_{n-\epsilon_1}, z_{n-\epsilon_2}, \ldots, z_n)^{\tau},$$

$$
\begin{aligned}
Z_{n+1} &= (z_{n+\epsilon_1}, z_{n+\epsilon_2}, ..., z_{n+1})^{\tau},\\
F_n &= (f_{n-\epsilon_1}, f_{n-\epsilon_2}, ..., f_n)^{\tau},\\
F_{n+1} &= (f_{n+\epsilon_1}, f_{n+\epsilon_2}, ..., f_{n+1})^{\tau},\\
H_n &= (\eta_{n-\epsilon_1}, \eta_{n-\epsilon_2}, ..., \eta_n)^{\tau},\\
H_{n+1} &= (\eta_{n+\epsilon_1}, \eta_{n+\epsilon_2}, ..., \eta_{n+1})^{\tau},\\
X_n &= (\xi_{n-\epsilon_1}, \xi_{n-\epsilon_2}, ..., \xi_n)^{\tau}.
\end{aligned}
$$

3.1. TRUNCATION ERROR

The linear difference operator $\mathcal{L}$ associated with the developed hybrid block method is expressed as:

$$
\begin{aligned}
\mathcal{L}[z(t_n);h] = \sum_j [&\phi_j z(t_n + jh) - h\xi_j z'(t_n + jh)\\
&- h^2\psi_j z''(t_n + jh) - h^3 \upsilon_j z'''(t_n + jh)],\\
&j = 0, r, s, 1-s, 1-r, 1,
\end{aligned} \tag{30}
$$

where $z(t_n)$ is an arbitrary function that is continuously differentiable on $[a, b]$. Expanding equation (30) as a Taylor series around t_n and grouping terms gives:

$$
\begin{aligned}
\mathcal{L}[z(t_n);h] = C_0 z(t_n) + C_1 h z'(t_n) + C_2 h^2 z''(t_n)\\
+ \ldots + C_p h^p z^{(p)}(t_n) + \ldots +\\
C_{p+2} h^{p+2} z^{(p+2)}(t_n).
\end{aligned} \tag{31}
$$

The vector $C_j, j = 0, 1, 2, \ldots, N$ are the coefficients. The method is said to be of order p if

$$C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0 \quad \text{and} \quad C_{p+2} \neq 0.$$

The coefficient C_{p+2} is referred to as the error constant. The method yields the following truncation error:

$$T_h = C_{p+2} h^{p+2} + O(h^{p+3}), \tag{32}$$

where T_h is the truncation error, and the order of the method is determined by the smallest non-zero coefficient C_{p+2}.

The truncation error for this method is determined as follows:

$$
\mathcal{L}[z(t_n);h] = \begin{cases}
\frac{\left(122\sqrt{3}+27\right)h^{10}z^{(10)}(t_n)}{31558780915200} + O(h^{11})\\
\frac{\left(27-122\sqrt{3}\right)h^{10}z^{(10)}(t_n)}{31558780915200} + O(h^{11})\\
\frac{\left(27-122\sqrt{3}\right)h^{10}z^{(10)}(t_n)}{31558780915200} + O(h^{11})\\
\frac{\left(122\sqrt{3}+27\right)h^{10}z^{(10)}(t_n)}{31558780915200} + O(h^{11})\\
\frac{h^{13}z^{(13)}(t_n)}{237342897792000} + O(h^{14})\\
\frac{\sqrt{\frac{1}{33}\left(9981-3704\sqrt{3}\right)}h^{10}z^{(10)}(t_n)}{95632669440} + O(h^{11})\\
-\frac{h^{10}\left(\sqrt{\frac{1}{33}\left(3704\sqrt{3}+9981\right)}z^{(10)}(t_n)\right)}{95632669440} + O(h^{11})\\
\frac{\sqrt{\frac{1}{33}\left(3704\sqrt{3}+9981\right)}h^{10}z^{(10)}(t_n)}{95632669440} + O(h^{11})\\
-\frac{h^{10}\left(\sqrt{\frac{1}{33}\left(9981-3704\sqrt{3}\right)}z^{(10)}(t_n)\right)}{95632669440} + O(h^{11})\\
\frac{h^{13}z^{(13)}(t_n)}{237342897792000} + O(h^{14})
\end{cases}
$$

The block method has $C_0 = C_1 = C_2 = \cdots = C_9 = 0$ and

$$
\begin{aligned}
C_{10} = [&7.5513 \times 10^{-12}, -5.8402 \times 10^{-12},\\
&-5.8402 \times 10^{-12}, 7.5513 \times 10^{-12}, 0,\\
&1.0869 \times 10^{-10}, -2.3308 \times 10^{-10},\\
&2.3308 \times 10^{-10}, -1.0869 \times 10^{-10}, 0]^{\tau}.
\end{aligned}
$$

This establishes that the method has order $p = 8$.

3.2. ZERO-STABILITY

The method stability are analyzed using equation (29). In this equation, as $h \to 0$ the expression becomes:

$$\Gamma_1 Z_{n+1} = \Gamma_0 Z_n. \tag{33}$$

The characteristic polynomial $\rho(\lambda)$ associated with this difference equation is defined as:

$$\rho(\lambda) = \det[\lambda\Gamma_1 - \Gamma_0]. \tag{34}$$

Substituting the appropriate matrices into the determinant, the characteristic polynomial is obtained as

$$-\lambda^4(\lambda + 1) = 0. \tag{35}$$

Solving this polynomial yields the roots

$$\lambda = 0, 0, 0, 0, -1.$$

Definition 1 (Singh and Ramos [14]). *A block method is zero stable if the roots $\lambda_i, i = 1, 2, ..., s$ of the first characteristic polynomial $\rho(\lambda)$ satisfy $|\lambda_i| \leq 1$, and for those roots with $|\lambda_i| = 1$, the multiplicity must not exceed two.*

This confirmed that the method is zero-stable.

3.3. CONSISTENCY

Definition 2 (Alkasassbeh and Omar [17]). *A block method is said to be consistent if it has order $p \geq 1$.*

Since the hybrid block method has an order of $p = 8$, this indicates that the method is consistent.

3.4. CONVERGENCE

Theorem 2 (Omar and Abdelrahim [7]). *The necessary and sufficient conditions for a numerical method to be convergent are that it must be consistent and zero-stable.*

Since the method meets both the conditions of zero-stability and consistency, we conclude that the method is convergent.

3.5. LINEAR STABILITY

The method stability is determined by applying equation (29) to the test equations $z' = \lambda z$, $z'' = \lambda^2 z$, and $z''' = \lambda^3 z$, where $\lambda \in \mathcal{R}$. By setting $\mu = \lambda h$, we obtain:

$$Z_{n+1} = \Delta(\mu) Z_n,$$

where

$$\Delta(\mu) = (\Gamma_1 - \mu^2 K_1 - \mu^3 P_1)^{-1} \cdot (\Gamma_0 + \mu N_0 + \mu^2 K_0 + \mu^3 P_0)$$

is the amplification matrix. The method's stability is based on the eigenvalues of the amplification matrix, as these determine the growth or decay of errors in the numerical solution. By analyzing the amplification matrix gives the following expression:

$$\rho(\mu) = -\frac{\begin{array}{l} 3(-221298739200 + \mu(-77989443840 \\ \quad + \mu(-31622088960 + \mu(-3027166272 \\ \quad + \mu(-1033303104 + \mu(-10959792 \\ \quad + \mu(-10494096 + \mu(363104 \\ \quad + \mu(-24680 + \mu(2099 + 47\mu)))))))))) \end{array}}{\begin{array}{l} 663896217600 - 14202397440\mu^2 \\ \quad + 164142720\mu^4 + \mu^6(-1386000 \\ \quad + \mu(22176 + \mu(-60 + \mu(1992 \\ \quad + \mu(-297 + 16\mu))))). \end{array}}$$

This expression represents the spectral radius of the method, which is a crucial part of the linear stability analysis.
This region of absolute stability is illustrated in Figure 1.

Definition 3. *Lambert [1] A numerical method is said to be A-stable if its region of absolute stability contains the entire negative complex half-plane.*

The stability region of the method lies entirely within the negative complex half-plane, this confirmed that the method is *A-stable*.

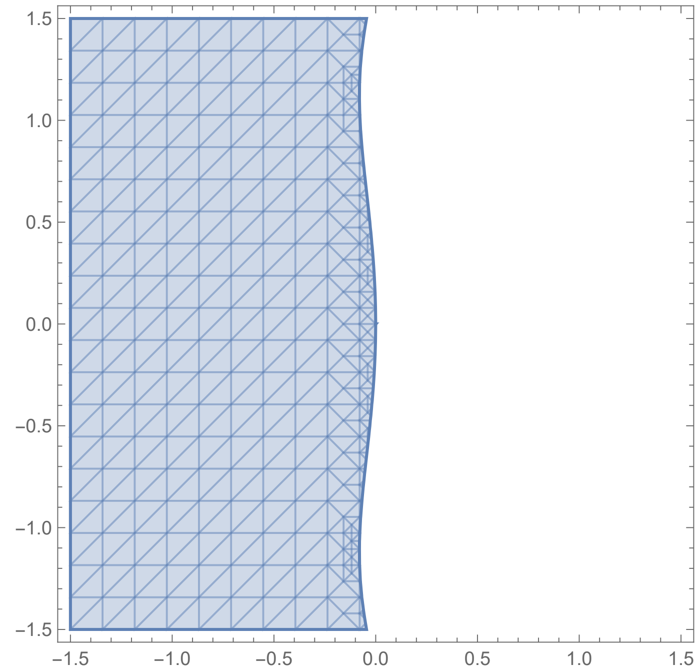

Figure 1. Stability region.

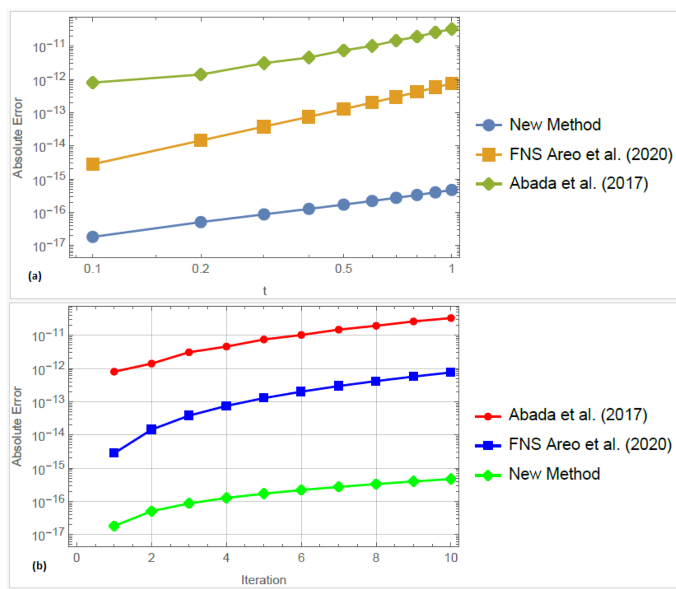

Figure 2. (a) Absolute errors, (b) Efficiency curves.

4. IMPLEMENTATION

In this section, the method is applied as single-step block numerical integrators to solve six test problems, enabling the simultaneous approximation of solutions $[z_{n+r}, ..., z_{n+1}]^\tau$ for n ranging from 0 to $N - 1$ across non-overlapping sub-intervals $[t_0, t_1], ..., [t_{N-1}, t_N]$.

Step 1: Initialize N, the total number of sub-intervals, and compute the constant step size $h = \frac{(b-a)}{N}$, ensuring $N > 0$ is a positive integer. For $n = 0$, determine the values of $[z_r, ..., z_1]^\tau$ simultaneously over the interval $[t_0, t_1]$ using the known value z_0 from the IVP as stated in Equation (1).

Step 2: For $n = 1$, simultaneously compute $[z_{1+r}, ..., z_2]^\tau$ over the interval $[t_1, t_2]$, using z_1 obtained from the previous block.

Step 3: Continue this process for $n = 2, 3, ..., N-1$, systematically obtaining approximate solutions for equation (1) over the sub-intervals $[t_2, t_3], \ldots, [t_{N-1}, t_N]$, ensuring that the sub-intervals remain non-overlapping.

The derivation, analysis, and implementation of the proposed method are performed using Mathematica 13.0 software, which provides an efficient computational framework for evaluating the accuracy and efficiency of the proposed schemes.

Example 1.
Consider a slightly stiff linear example solved by Singh *et al.* [18] and Singla *et al.* [19]:

$$z'' - z = 0, \quad z(0) = 0, \quad z'(0) = -1, \quad h = \frac{1}{10}.$$

The exact solution is given by

$$z(t) = 1 - e^t.$$

Example 2.
Consider the nonlinear stiff problem solved by [20, 24]:

$$z'' - 2z^3 = 0, \quad z(1) = 1, \quad z'(1) = -1, \quad h = 0.1.$$

The exact solution is given by

$$z(t) = \frac{1}{t}.$$

Example 3.
Consider the linear second-order ODE describing the instantaneous charge $z(t)$ on the capacitor in an LRC series circuit

$$Lz'' + Rz' + (\frac{1}{C})z = E(t), \quad z(0) = 0, \quad i(0) = 0,$$

where the letters L, C, and R are the inductance, capacitance, and resistance respectively. The impressed voltage is given by $E(t)$, and $i(t)$ is the current. We solve the problem with the following parameters: $L = 1, R = 20, C = 0.005, E(t) = 150$, and $h = 0.01$ (see [21]).
The exact solution is given by $z(t) = \frac{3}{4}(1 - e^{-10t}(\cos(10t)+\sin(10t)))$.

Example 4.
Real-life problem: Cooling of a body.
The temperature z (in degrees) of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2z}{dt^2} + \frac{dz}{dt} = 0$. Using the substitution $q = \frac{dz}{dt}$ or otherwise, determine z in terms of t, given that $z = 60$ when $t = 0$ and $z = 35$ when $t = 6$. Additionally, find the time (to the nearest minute) at which the rate of cooling of the body falls below one degree per minute. Problem like this are also explored in [22, 23].
Formulation of the problem:

$$z'' + \frac{z'}{3} = 0, \ z(0) = 60, \ z'(0) = -\frac{80}{9}, \ h = 0.1.$$

The exact solution is given by $z(t) = \frac{80}{3}e^{-\frac{1}{3}t} + \frac{100}{3}$.

Example 5.

Table 1. **Comparison of absolute errors for Example 1.**

t	Adaba *et al.* [18]	Areo *et al.* [19]	New method
0.1	8.0000×10^{-13}	2.8033×10^{-15}	1.8200×10^{-17}
0.2	1.4000×10^{-12}	1.4461×10^{-14}	5.0800×10^{-17}
0.3	3.0700×10^{-12}	3.8025×10^{-14}	8.6800×10^{-17}
0.4	4.5500×10^{-12}	7.4663×10^{-14}	1.2660×10^{-16}
0.5	7.4300×10^{-12}	1.2801×10^{-13}	1.7050×10^{-16}
0.6	1.0180×10^{-11}	1.9995×10^{-13}	2.1910×10^{-16}
0.7	1.4710×10^{-11}	2.9532×10^{-13}	2.7280×10^{-16}
0.8	1.9190×10^{-11}	4.1633×10^{-13}	3.3220×10^{-16}
0.9	2.5980×10^{-11}	5.6954×10^{-13}	3.9780×10^{-16}
1	3.2810×10^{-11}	7.5828×10^{-13}	4.7020×10^{-16}

Table 2. **Comparison of absolute errors for Example 2.**

t	New method	Error in Abdelrahim *et al.* [20]	Error in Yahaya and Sagir [24]
1.1	1.3884×10^{-9}	1.6603×10^{-10}	1.3736×10^{-6}
1.2	4.4753×10^{-9}	8.2396×10^{-10}	1.5545×10^{-6}
1.3	8.7109×10^{-9}	6.8322×10^{-7}	2.2569×10^{-6}
1.4	1.3849×10^{-8}	1.3895×10^{-6}	2.3805×10^{-6}
1.5	1.9790×10^{-8}	2.2801×10^{-6}	2.6336×10^{-6}

Consider the Van der Pol oscillator, governed by the second-order differential equation

$$z'' - 2\mu(1 - z^2)z' + z = 0, \ z(0) = 0, \ z'(0) = 0.5,$$

where $\mu = 0.005$. Here, z is a function of time t, and μ represents the nonlinearity and strength of damping. Since this problem does not have an analytical solution, it is solved numerically using a step size of $h = 0.1$ (see [3]).

Example 6.
Consider the Duffing equation, a nonlinear second-order differential equation that models systems with a nonlinear restoring force, given by:

$$z'' + \delta z' - \beta z + z^3 = 0, \quad z(0) = 0.5, \quad z'(0) = 0,$$

where δ and β are constants, and the nonlinear term z^3 represents the cubic stiffness. This equation does not have a simple closed-form analytical solution due to its nonlinear nature, making it ideal for numerical methods. This example is solved with $\beta = 1$ and $\delta = 0.2$, using a step size of $h = 0.01$ (see [25]).

5. RESULTS AND DISCUSSION

This section presents the numerical results obtained by applying the proposed method to a selection of benchmark problems, previously addressed in the literature. The primary objective is to evaluate and demonstrate the enhanced accuracy, stability, and overall performance of the proposed method.

The numerical results presented in Table 1 highlight the superior accuracy of the new method compared to the hybrid block methods developed by Adaba *et al.* [18] and Areo *et al.* [19]. This accuracy advantage is clearly illustrated by the error and efficiency curves shown in Figure 2.

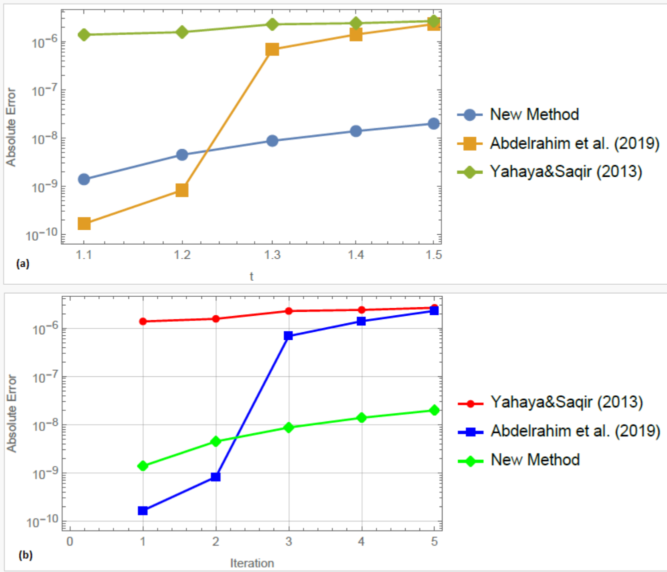

Figure 3. Comparison of (a) Absolute errors and (b) Efficiency curves.

Table 3. Comparison of absolute errors for Example 3.

t	New method	Error in Jator and Li [21]
0.0	0.0000	0.0000
0.1	7.635709×10^{-9}	161.8373×10^{-5}
0.2	5.922117×10^{-9}	111.2906×10^{-5}
0.3	4.375127×10^{-10}	35.23520×10^{-5}
0.4	1.369370×10^{-9}	225.1629×10^{-5}
0.5	9.363970×10^{-11}	281.2812×10^{-5}

In Table 2, the new method shows enhanced performance, with improved accuracy as the iterations progress, compared to the two-step third-derivative hybrid block method developed by Abdelrahim *et al.* [20] and the three-step hybrid block method proposed by Yahaya and Sagir [24]. Figure 3 illustrates the absolute errors and efficiency curves for all methods, highlighting the differences in performance.

In Table 3, the new method demonstrates a significant improvement in accuracy compared to the four-step block method developed by Jator and Li [21]. This improvement is further highlighted by the error and efficiency curves shown in Figure 4.

The results presented in Table 4 compare the absolute errors of the newly developed method with those of the methods proposed by Omole and Ogunware [22] and Obarhua [23]. To further highlight the accuracy of the new method, the error and efficiency curves are presented in Figure 5.

The results for the Van der Pol oscillator in Example 5 highlight the effectiveness of the proposed method. As shown in Table 5 and Figure 6, the solution's accuracy demonstrates that the one-step optimized hybrid block method outperforms well-established methods from the literature.

The plot in Figure 7 illustrates the numerical solution for Example 6, computed over the interval $[0, 2]$ with a step size of

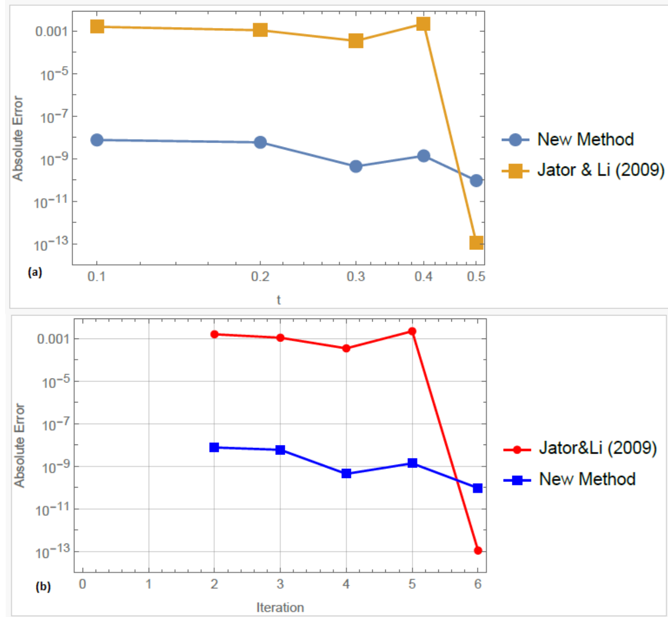

Figure 4. Comparison of (a) Absolute errors and (b) Efficiency curves.

Table 4. Comparison of absolute errors for Example 4.

t	Error in Omole and Ogunware [22]	Error in Obarhua [23]	New method
0.1	3.5500×10^{-11}	2.344791×10^{-13}	7.1230×10^{-15}
0.2	4.5800×10^{-11}	2.202682×10^{-13}	1.4887×10^{-14}
0.3	7.0000×10^{-11}	3.935749×10^{-12}	2.2397×10^{-14}
0.4	6.5000×10^{-12}	2.704951×10^{-12}	2.9660×10^{-14}
0.5	3.3300×10^{-11}	7.599112×10^{-12}	3.6686×10^{-14}
0.6	4.2000×10^{-11}	1.569518×10^{-12}	4.3481×10^{-14}
0.7	4.3800×10^{-11}	2.756872×10^{-12}	5.0053×10^{-14}
0.8	1.0700×10^{-10}	4.375392×10^{-11}	5.6410×10^{-14}
0.9	6.5800×10^{-11}	6.474571×10^{-11}	6.2558×10^{-14}
1	1.6900×10^{-10}	9.100178×10^{-11}	6.8505×10^{-14}

Table 5. Comparison of numerical results for Example 5.

t	New method	Allogmany *et al.* [3]	NDSolve
0.0	0.0000	0.0000	0.0000
0.1	0.0499416	0.0500419	0.0500417
0.5	0.240307	0.242703	0.242704
1.3	0.495997	0.496911	0.496936
2.1	0.458881	0.453153	0.453212
3.1	0.021110	0.0227339	0.0227741
4.1	−0.444496	−0.450239	−0.45029
5.1	−0.511088	−0.521028	−0.521174

$h = 0.01$. The solution shows rapid changes near $t = 2$, indicating the stiffness of the Duffing equation, where variations in the independent variable or parameters become more pronounced. This stiffness presents challenges for numerical computation.

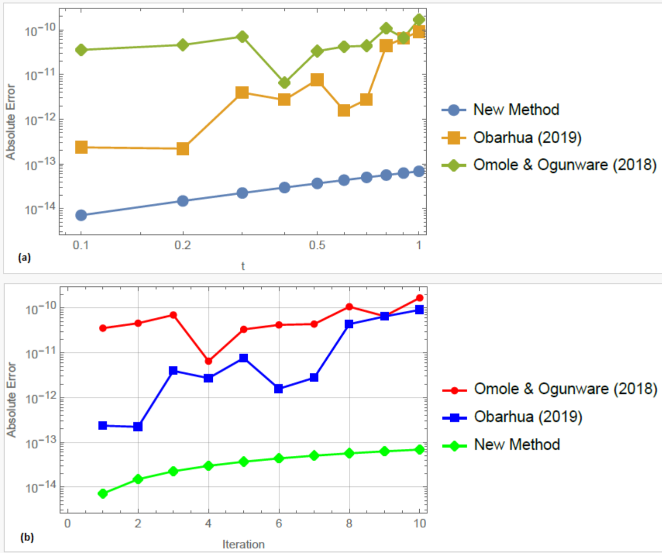

Figure 5. Comparison of (a) Absolute errors and (b) Efficiency curves.

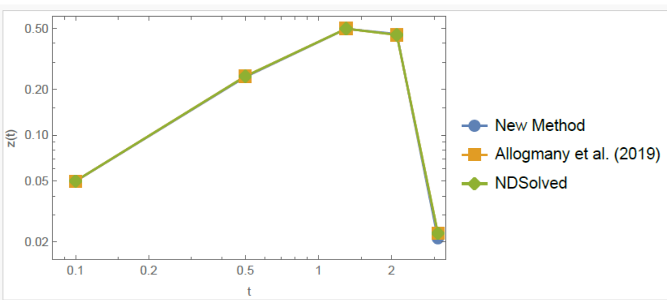

Figure 6. Graphical solution for the Van der Pol oscillator in Example 5.

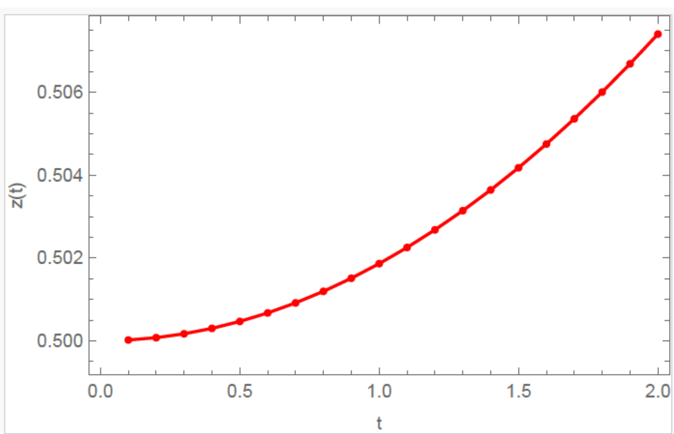

Figure 7. Approximation solution for Example 6.

6. CONCLUSION

This study introduces a one-step optimized hybrid block method designed specifically for the direct solution of general second-order ODEs. A thorough numerical analysis confirms that the method possesses essential properties, including zero-stability, consistency, convergence, and *A*-stability. These characteristics make the method well-suited for solving both stiff and non-stiff ODEs.

Extensive numerical simulations were performed on a diverse set of test problems related to second-order IVPs. The results, presented in Tables 1 to 5 and Figures 2 to 7, demonstrate the method's high accuracy. The integration of optimization techniques significantly enhanced both the accuracy and stability when solving second-order ODEs. The computational cost of the novel one-step method is comparable to or even lower than that of existing methods, despite incorporating higher-order derivative evaluations and optimization steps. This efficiency arises from several factors: reduced function evaluations, robust stability properties that permit larger step sizes, and optimized coefficients that minimize redundant computations. Although the per-step effort is slightly higher, the overall cost is offset by the need for fewer total steps and the improved stability of the method. Quantitative comparisons, such as wall-clock time or number of floating-point operations (FLOPs), could further substantiate these cost advantages.

In conclusion, the newly developed method proves to be computationally reliable and effective for solving general second-order IVPs, providing notable advancements over existing numerical methods. All the proposed schemes in this paper are specifically designed to effectively address the defined problem of study, i.e., second-order ODEs. However, other researchers might attempt to apply these methods to higher-order ODEs in their current form or by making appropriate modifications.

DATA AVAILABILITY

We do not have any research data outside the submitted manuscript file.

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