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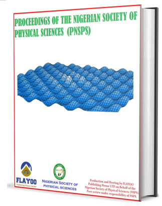

Characterization of quasi idempotent in semigroups of full contraction mappings

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ABSTRACT

Let $\mathcal{T}_n$ be the semigroup of full transformation of a finite set and let $X_n = \{1, 2, \cdots, n\}$. A transformation $\alpha : \mathrm{Dom}\,\alpha \subset X_n \to \mathrm{Im}(\alpha) \subset X_n$ is said to be full or total transformation if $\mathrm{Dom}\,\alpha = X_n$. A transformation $\beta \in \mathcal{CT}_n$ is said to be full contraction mapping if $\forall\, x, y \in \mathrm{Dom}\,\alpha$, $|x\alpha - y\alpha| \leq |x - y|\, \forall\, x, y \in \mathrm{Dom}\,\alpha$. Let $\mathcal{CT}_n$ be the semigroup of full contraction transformation and let $\mathcal{QCT}_n$ be its quasi-idempotent of full contraction transformation of X_n. In this paper, we characterized quasi-idempotent elements into matching blocks, non-matching blocks and self matching blocks of $\mathcal{CT}_n$ and later come up with a strong and useful theorem.

Keywords: Semigroup, Contraction mapping, Quasi-idempotent.

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1. INTRODUCTION

Let $X_n = \{1, 2, \ldots, n\}$. A (partial) transformation $\alpha : \mathrm{Dom}\,\alpha \subseteq X_n \to \mathrm{Im}\,\alpha \subseteq X_n$ is said to be *full* or *total* if $\mathrm{Dom}\,\alpha = X_n$; otherwise it is called *strictly* partial. The *fix* of α is denoted and defined by $F(\alpha)$, where

$$| F(\alpha) | = | \{x \in \mathrm{Dom}\,\alpha : x\alpha = x\} | . \quad (1)$$

Let $X_n = \{1, 2, \cdots n\}$ be a finite set and let $\mathcal{T}_n$ and $\mathcal{CT}_n$ be a full transformation semigroup and a semigroup of full contraction mapping on X_n respectively. A transformation $\alpha : \mathrm{Dom}\,\alpha \subset X_n \to Im(\alpha) \subset X_n$ is said to be full or total transformation if $\mathrm{Dom}\,\alpha = X_n$. The set of full transformation of X_n, denoted by $\mathcal{T}_n$ more commonly known as the full transformation semigroup is also known as the full symmetric semigroup with binary composition of mapping as the semigroup operator defined on it. Adeshola and Umar [1] defined a transformation $\alpha \in \mathcal{CT}_n$ to be full contraction mapping if $\forall\, x, y \in \mathrm{Dom}\,\alpha$,

$$|x\alpha - y\alpha| \leq |x - y|\, \forall\, x, y \in \mathrm{Dom}\,\alpha. \quad (2)$$

The theory of finite semigroups has been of importance in theoretical computer science since the 1950s because of the link between finite semigroup and finite automata via the syntactic monoid. Another certain importance and interesting concept such as the concept of idempotent rank, nilpotent rank, etc, developed largely independently. Some years back, Garba *et al.* [2–4] worked on products of idempotent and quasi-idempotent of transformation semigroup. In the past few years, Adeshola and Umar [1] worked on combinatorial results for certain semigroup of full contraction mapping of a finite chain which is another interesting aspect of semigroup where much research have not been carried-out. See Online Encyclopedia of Integer Sequence [7].

An element α of semigroup S is an idempotent if $\alpha = \alpha^2$. Clearly, a full transformation semigroup α is idempotent if and

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only if

$$F(\alpha) = \mathrm{Im}(\alpha), \tag{3}$$

where $F(\alpha)$ is the set of all fixed point of the transformation and $\mathrm{Im}(\alpha)$ is the image set of the transformation.

The fix of a transformation β is defined and denoted by

$$F(\alpha) = |F(\beta)| = x \in \mathrm{Dom}(\beta) : x\beta. \tag{4}$$

Let X_n be the finite set $\{1, 2 \cdots n\}$ and $\mathcal{T}_n$ be the full transformation semigroup on X_n. Then a transformation $\beta \in \mathcal{T}_n$ will be called quasi idempotent if β is not an idempotent but β^2 is an idempotent. That is, $\beta \neq \beta^2 = \beta^4$, where

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}. \tag{5}$$

Note that β is not an idempotent, but

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}.$$

Similarly,

$$\beta^4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}.$$

Therefore, it shows that $\beta^2 = \beta^4$.

In this section, we introduce basic terminologies and definition of terms. In Section 2 we adopt the methodology involved in obtaining the results as well as classification through characterization, the matching blocks, non-matching blocks and self-matching blocks. In Section 3 we present, analyze and discuss the results, and finally, Section 4 deals with the Conclusion.

Let β be a map in $\mathcal{CT}_n$. We define a matching pair of β to be a pair (A_i, A_j) of blocks of β for which $A_i\beta \in A_j$ and $A_j\beta \in A_i$. The blocks A_i and A_j will be referred to as matching blocks of β. If $i = j$, then we will say that (A_i, A_j) is a self matching pair and A_i is a stationary block of β. For standard concepts in semigroup and transformation semigroup theory, see for example Ref. [5].

2. METHODOLOGY AND CHARACTERIZATION OF QUASI-IDEMPOTENT SEMIGROUP OF FULL CONTRACTION MAPPINGS: $\mathcal{QCT}_N$

The method of research adopted in this article is by reviewing necessary and relevant papers on the theory of algebraic properties of semigroup of full transformation contraction mappings, sub-semigroups generated by quasi-idempotents in certain finite semigroups of mappings, please see Garba *et al.* [2–4]. The work done is based on characterization of quasi-idempotents in $\alpha \in \mathcal{CT}_n$. In this section, we present a set of theoretic characterization of quasi-idempotent in $\alpha \in \mathcal{CT}_n$, as matching blocks, non-matching blocks and self matching blocks. This will enable us to identify them among other elements of $\alpha \in \mathcal{CT}_n$.

Matching Block

As defined earlier, let β be a map in $\mathcal{CT}_n$. A matching pair of β to be a pair (A_i, A_j) of blocks of β for which $A_i\beta \in A_j$ and $A_j\beta \in A_i$. The blocks A_i and A_j will be referred to as matching blocks of β. For example, let

$$\beta = \begin{pmatrix} \{1,5\} & \{2,3,4\} \\ 4 & 5 \end{pmatrix}. \tag{6}$$

We can observe that $A_i\beta \in A_j$ and $A_j\beta \in A_i$. Therefore; the map β is a matching block.

Non-matching Blocks

As defined earlier, let β be a map in $\mathcal{CT}_n$. We define a non-matching pair of β to be a pair (A_i, A_j) of blocks of β for which $A_i\beta \notin A_j$ but $A_j\beta \in A_i$ or $A_i\beta \in A_j$ but $A_j\beta \notin A_i$. The blocks A_i and A_j will be referred to as non-matching blocks of β. For example, consider

$$\beta = \begin{pmatrix} \{1,2\} & \{3,4,5\} \\ 4 & 5 \end{pmatrix}, \tag{7}$$

we observe that $A_i\beta \in A_j$ but $A_j\beta \notin A_i$. Also consider

$$\beta = \begin{pmatrix} \{4,5\} & \{2,3\} \\ 5 & 4 \end{pmatrix}, \tag{8}$$

we observe that $A_i\beta \notin A_j$ but $A_j\beta \in A_i$.

Self Matching Blocks

As defined earlier, if

$$i = j, \tag{9}$$

then we will say that (A_i, A_j) is a self matching block and A_i is a stationary block of β.

$$\beta = \begin{pmatrix} \{5,6\} \\ 5 \end{pmatrix}. \tag{10}$$

It follows that

$$A_i = A_j. \tag{11}$$

Therefore, β is a self matching block.
$\Rightarrow$; A_i is a stationary block.

3. PRESENTATION AND DISCUSSION OF RESULTS

In this section, we present the theoretic characterization of quasi-idempotent in $\mathcal{CT}_n$, this will enable us to identify them among other elements of $\mathcal{CT}_n$. To begin with, we have the following definitions.

Let β be a map in $\mathcal{CT}_n$. We define a matching pair of β to be a pair (A_i, A_j) of blocks of β for which $A_i\beta \in A_j$ and $A_j\beta \in A_i$. The blocks A_i and A_j will be referred to as matching blocks of β. If $i = j$, then we will say that (A_i, A_j) is a self matching pair and A_i is a stationary block of β.

Lemma 3.1. *If A_i is a matching block of $\beta \in \mathcal{CT}_n$, then A_i is a stationary block of β^2.*

Proof. Suppose (A_i, A_j) is a matching pair of blocks of β. Then $A_i\beta^2 = (A_i\beta) \in A_j\beta \in A_i$ showing that A_i is a stationary block of β^2.

Example 1:

$$\beta = \begin{pmatrix} \{1,3\} & \{2,4\} \\ 2 & 3 \end{pmatrix},$$

$$\beta^2 = \begin{pmatrix} \{1,3\} & \{2,4\} \\ 3 & 2 \end{pmatrix}.$$

Example 2:

$$\beta = \begin{pmatrix} \{1,5\} & \{2,3,4\} \\ 4 & 5 \end{pmatrix},$$

$$\beta^2 = \begin{pmatrix} \{1,5\} & \{2,3,4\} \\ 5 & 4 \end{pmatrix}.$$

Lemma 3.2. *if $\beta \in \mathcal{CT}_n$ consists only of matching blocks(not all stationary), then β is a quasi-idempotent.*

Proof. Suppose $\beta \in \mathcal{CT}_n$ is a map containing only matching pair of blocks and let A_i be an arbitrary block of β; A_i is a matching block of β and by Lemma 3.1, A_i is a stationary block of β^2. Thus β^2 is an idempotent.

Example 3:

$$\beta = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 3 & 2 \end{pmatrix},$$

$$\beta^2 = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 2 & 3 \end{pmatrix},$$

$$\beta^4 = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 2 & 3 \end{pmatrix},$$

$\Rightarrow \beta^2 = \beta^4$

Example 4:

$$\beta = \begin{pmatrix} \{1,4,5\} & \{2,3\} \\ 3 & 4 \end{pmatrix},$$

$$\beta^2 = \begin{pmatrix} \{1,4,5\} & \{2,3\} \\ 4 & 3 \end{pmatrix},$$

$$\beta^4 = \begin{pmatrix} \{1,4,5\} & \{2,3\} \\ 4 & 3 \end{pmatrix},$$

$\Rightarrow \beta^2 = \beta^4$.

Lemma 3.3. *Every quasi-idempotent $\beta \in \mathcal{CT}_n$ must contain at least one matching pair of blocks (either self – matching or non-stationary).*

Proof. Suppose $\beta \in \mathcal{CT}_n$ is a map containing no matching pair of blocks and let A_i be any of its blocks. Then there exists a block $A_j (j \neq i)$ for β for which $A_i\beta \in A_j$ and $A_j\beta \neq A_i$. Therefore, $A_i\beta^2 = (A_i\beta)\beta \in A_j\beta \notin A_i$ implying that A_i is not fixed by β^2 and so β^2 is not an idempotent, contradicting the choice of β as a quasi-idempotent. Hence β must contain at least one matching pair of blocks.

Example 5:

$$\beta = \begin{pmatrix} \{1\} & \{2,3,4\} \\ 3 & 4 \end{pmatrix},$$

Example 6:

$$\beta = \begin{pmatrix} \{1\} & \{2\} & \{3\} & \{4\} \\ 2 & 3 & 4 & 5 \end{pmatrix},$$

then we have the following theorem:

Theorem 3.4. *Let $\beta \in \mathcal{CT}_n$, then β is a quasi-idempotent if and only if the image of each non-matching block of β is contained in a matching block of β(or stationary).*

Proof. If $\beta \in \mathcal{CT}_n$ contains no non-matching blocks, then the argument is trivial by Lemma 3.2. Now suppose $\beta \in \mathcal{CT}_n$ is a map consisting of both matching and non-matching blocks in which every non-matching block is mapped into a matching block, we shall show that β is a quasi-idempotent.

Let

$$\beta = \begin{pmatrix} A_1 & \cdots & A_r & A_{r+1} & A_{r+2} & \cdots & A_{r+s} & B_1 & \cdots & B_t \\ a_1 & \cdots & a_r & a_{r+2} & a_{r+1} & \cdots & a_{r+s-1} & b_1 & \cdots & b_t \end{pmatrix}, \quad (12)$$

where $A_1 \cdots A_r$ are stationary blocks of $\beta, A_{r+1} \cdots A_{r+s}$ are matching blocks of β and $B_1 \cdots B_t$ are non matching blocks of β. Also $a_i \in A_i (1 \leq i \leq r+s)$ and each $b_j (1 \leq j \leq t)$ belongs to one of $A_i's$. Observe that $A_i\beta^2 = a_i$ for each $i = 1, \cdots, r+s$ and if $b_j \in A_i$, then $(A_i \cup B_j)\beta^2 = a_i$. Therefore β can be expressed as

$$\beta^2 = \begin{pmatrix} C_1 & \cdots & C_u & A_{u+1} & \cdots & A_r & C_{r+1} & \cdots & C_{r+v} & A_{r+v+1} & \cdots & A_{r+s} \\ a_1 & \cdots & a_u & a_{u+1} & \cdots & a_r & a_{r+1} & \cdots & a_{r+v} & a_{r+v+1} & \cdots & a_{r+s} \end{pmatrix}, \quad (13)$$

with the assumption that $A_1, \cdots, A_u$ and $A_{r+1}, \cdots, A_{r+v}$ are respectively the stationary and matching blocks of β containing $b_j's (j = 1, \cdots, t)$ and for each $i = 1, \cdots, u, r+1, \cdots, C_1 = A_i \cup (U_j B_j)$. The second union runs over each j for which $b_j \in A_i$. Then it is clear that β^2 is an idempotent and β is a quasi- idempotent.

Conversely, suppose β is a quasi-idempotent. Then by Lemma 3.3 β must contain at least one matching pair of blocks. If all its blocks are matching, there is nothing to prove. Therefore let β contains $A_i (1 \leq i \leq s)$ and $B_j (1 \leq j \leq t)$ matching and non-matching blocks respectively. Let us further suppose, by way of contradiction, that for some j, $B_j\beta \in A_i$ for all i. Then either $B_j\beta^2 \in A_i$ for some i or $B_j\beta^2 \notin A_i$ for all i. In the former, since $B_j \notin A_i, B_j$ is not fixed by β^2. In the latter, $B_j\beta^2 \notin B_k$ for some $k = j$ and again B_j is not fixed by β^2. Thus, in both cases β^2 is not an idempotent contradicting the choice of β as a quasi–idempotent.

3.1. DISCUSSION OF RESULTS

As our study reads, characterization of quasi idempotent of semigroup of full contraction mappings, is a new class of semigroup where much work has not been done. We classified elements under quasi idempotent of semigroup of full contraction mappings into maching blocks, non-maching blocks and self matching blocks. Lemma 3.1 solves problem on matching blocks, Lemma 3.2 explains only matching blocks not all stationary while Lemma 3.3 provides solutions on at least one matching pair of blocks (either self-matching or non-stationary). On this note, it is important to note that in the course of study, we came across a feature where the image of each non-matching block of β is contained in a matching block of β (or stationary). This is exactly the consequence of equations (12) and (13) of Theorem 3.4.

4. CONCLUSION

Sequel to the results obtained, we can conclude that the contraction mappings have shown to be an important area of study in the theory of transformation semigroups and this research has shown how to characterize a quasi-idempotent elements in semigroup of full contraction mappings. Also it is an interesting sector of transformation semigroups and much research work can still be done. As earlier mentioned, semigroups of full contraction mappings is a new class of transformation semigroups and much research work has not been done. In this work we have only studied the quasi idempotent of contraction while other sub-semigroups can still be studied (where applicable). In conclusion, contraction mappings have appeared to be promising area to study in the theory of transformation semigroups.

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