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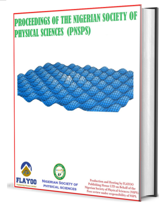

Estimates for a class of analytic and univalent functions connected with quasi-subordination

Rasheed Olawale Ayinla[a,*], Ayotunde Olajide Lasode[b]

[a]*Department of Mathematics and Statistics, Kwara State University, PMB 1530, Malete, Nigeria*
[b]*Department of Mathematics, University of Ilorin, PMB 1515, Ilorin, Nigeria*

ABSTRACT

Geometric Function Theory, an active field of study that has its roots in complex analysis, has gained an impressive attention from many researchers. This occurs largely because it deals with the study of geometric properties of analytic (and univalent) functions where many of its applications spread across many fields of mathematics, mathematical physics and engineering. Notable areas of application include conformal mappings, special functions, orthogonal polynomials, fluid flows in physics, and engineering designs. The investigations in this paper are on a subclass of analytic and univalent functions defined in the unit disk Ω and denoted by $\mathbb{Q}_q^\alpha(m)$. The definition of the new class encompasses some well-known subclasses of analytic and univalent functions such as the classes of starlike functions, Yamaguchi functions, and Ma-Minda functions. Two key mathematical principles involved in the definition of the class are the principles of Taylor's series and quasi-subordination. Some of the investigations carried out on functions $f \in \mathbb{Q}_q^\alpha(m)$ are however, the upper estimates for some initial bounds, the solution to the well-known Fekete-Szegö problem and the upper estimate for a Hankel determinant.

Keywords: Starlike function, Ma-Minda function, Yamaguchi function, Fekete-Szegö functional.

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1. INTRODUCTION

Let $\mathbb{A}$ denote the class of complex-valued functions f which are analytic in the open unit disk

$$\left\{z \mid z \in \mathbb{C} \text{ and } |z| < 1\right\} = \Omega.$$

Let $\mathbb{S}$ be a subclass of $\mathbb{A}$ and consists of functions of the Taylor's series form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_rz^r + \cdots \quad (f(0) = f'(0) - 1 = 0,\ z \in \Omega). \tag{1}$$

Class $\mathbb{S}$ is known as the class of normalized analytic and univalent functions defined in Ω. Some known subclasses of $\mathbb{S}$ are the classes of starlike functions (whenever $\mathbb{R}(zf'/f) > 0$), convex functions (whenever $\mathbb{R}[1 + (zf''/f')] > 0$), close-to-convex functions (whenever $\mathbb{R}(zf'/g) > 0$, g is starlike), close-to-star functions (whenever $\mathbb{R}(f/g) > 0$, g is starlike), bounded turnng functions (whenever $\mathbb{R}(f') > 0$), Yamaguchi functions (whenever $\mathbb{R}(f/z) > 0$) and many more. See Refs. [1–3] for more details.

The foundation of coefficient problems in the theory of normalized analytic and univalent functions of the series form in Eq. (1) can be ascribed to Bieberbach's conjecture [1, 4] of 1916 where it was conjectured that $|a_r| \leqslant r$, $\forall r \in \{2, 3, 4, \ldots\}$. In Ref.

*Corresponding Author Tel. No.: +234-803-4433-419.
e-mail: rasheed.ayinla@kwasu.edu.ng (Rasheed Olawale Ayinla)

[4], Duren affirmed that *coefficient problem* is the determination of that part of the $(n-1)$-dimensional complex plane that are occupied by the points $(a_2, a_3, a_4, \ldots, a_r)$ of function f. In 1985, Branges [5] validated the conjecture to be true and this affirmation elevated the field of Geometric Function Theory to one of the ever growing areas of possible research. Some coefficient problems that have been persistently studied include the upper estimates for coefficient bounds, Fekete-Szegö functional, Hankel determinants and Toeplitz determinants. Some recent studies in this regard are accessible in Refs. [6–12].

For any two analytic functions f and h, the function f is *subordinate* to h, expressed as

$$f(z) \prec h(z) \quad (z \in \Omega)$$

if there exists an analytic function

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots + \omega_r z^r + \cdots \\ (|\omega(z)| < 1,\ \omega(0) = 0,\ z \in \Omega), \tag{2}$$

such that

$$f(z) = h(\omega(z)).$$

Suppose h is a univalent function defined in Ω, then

$$f(z) \prec h(z) \quad \text{if and only if} \quad f(0) = h(0) \quad \text{and} \quad f(\Omega) \subset h(\Omega).$$

Robertson [13] introduced the notion of quasi-subordination as follows. For two analytic functions f and h, the function f is *quasi-subordinate* to h, expressed as

$$f(z) \prec_q h(z) \quad (z \in \Omega) \tag{3}$$

if there exist the analytic functions ω in Eq. (2) and

$$\varpi(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots + A_r z^r + \cdots \\ (|\varpi(z)| \le 1,\ z \in \Omega), \tag{4}$$

such that

$$f(z) = \varpi(z) h(\omega(z)) \quad (z \in \Omega).$$

According to MacGregor [14], suppose $\varpi(z) = 1$, then $f(z) = h(\omega(z))$, so that $f(z) \prec h(z)$ $(z \in \Omega)$. Also, if $\omega(z) = z$, then $f(z) = \varpi(z)h(z)$ and it is said that f is *majorized* by h, expressed as $f(z) \ll h(z)$. Note that quasi-subordination is a generalization of both subordination and majorization concepts. For some related works on quasi-subordination, see Refs. [15–17].

Using the notion of subordination, Ma and Minda [18] defined and studied the classes

$$\mathbb{S}^*(m) = \left\{ f : f \in \mathbb{A},\ \frac{zf'(z)}{f(z)} \prec m(z),\ \text{and } z \in \Omega \right\}$$

and

$$\mathbb{K}(m) = \left\{ f : f \in \mathbb{A},\ 1 + \frac{zf''(z)}{f'(z)} \prec m(z),\ \text{and } z \in \Omega \right\},$$

where

$$m(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots + B_r z^r + \cdots \\ (B_1 \in \mathbb{R}^+,\ z \in \Omega) \tag{5}$$

is an analytic function with positive real part in Ω, starlike with respect to 1, and symmetric with respect to the real axis; so that $m(0) = 1$ and $m'(0) > 0$. The function $m(z)$ has been found to amalgamate several functions whose real parts are positive. Some recent instances are present in Refs. [6, 7, 9, 11, 16, 17, 19] and many more.

In 1966, Pommerenke [12] demonstrated that the t-th Hankel determinant for the coefficients of function f in Eq. (1) is

$$\Delta_{t,r}(f) = \begin{vmatrix} 1 & a_{r+1} & \cdots\cdots & a_{r+t-1} \\ a_{r+1} & a_{r+2} & \cdots\cdots & a_{r+t} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r+t-1} & a_{r+t} & \cdots\cdots & a_{r+2(t-1)} \end{vmatrix}, \tag{6}$$

where $a_1 = 1$, $t \ge 1$, and $r \ge 1$. This determinant has been considered for specific values of r and t by many authors. For instance, see Refs. [8, 9, 11, 19]. It has also been established that the Fekete-Szegö functional

$$\nabla(\gamma, f) = |a_3 - \gamma a_2^2| \tag{7}$$

and the Hankel determinant in Eq. (6) are related such that

$$|\Delta_{2,1}(f)| = |a_3 - a_2^2| = \nabla(1, f).$$

For some recent works on Fekete-Szegö functional in Eq. (7), see Refs. [2, 6–8, 10, 11].

2. LEMMAS

The following lemmas shall be needed to prove our results. Firstly, let $\mathbb{P}$ denote the class of Carathéodory functions of the series type

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_r z^r + \cdots \quad (z \in \Omega) \tag{8}$$

which are analytic and satisfy the conditions: $p(0) = 1$ and $\mathbb{R}\, p(z) > 0$.

Lemma 2.1 ([1, 3])**.** *Let $p \in \mathbb{P}$. Then $|c_r| \le 2$, $\forall r \in \mathbb{N}$.*

Lemma 2.2 ([20])**.** *Let $p \in \mathbb{P}$. Then for any real number λ,*

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \le \begin{cases} 2(1-\lambda) & \text{if} \quad \lambda \le 0 \\ 2 & \text{if} \quad 0 \le \lambda \le 2 \\ 2(\lambda - 1) & \text{if} \quad \lambda \ge 2 \end{cases}$$

and for any complex number λ,

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \le 2 \max\{1, |1-\lambda|\}.$$

3. MAIN RESULTS

Definition 3.1. A function $f \in \mathbb{A}$ is said to be a member of class $\mathbb{Q}_q^\alpha(m)$ if the geometric condition

$$\left(\frac{zf'(z)}{f(z)} \right) \left(\frac{f(z)}{z} \right)^\alpha - 1 \prec_q (m(z) - 1) \tag{9}$$

holds for $\alpha \ge 0$, m in Eq. (5) and $z \in \Omega$.

Remark 3.2. The following are some of the subclasses of the new class $\mathbb{Q}_q^\alpha(m)$.

1. If $\alpha = 0$, then the geometric expressions on the LHS of Eq. (9) will reduce to the geometric expression for starlike functions discussed in Refs. [1, 3].
2. If $\alpha = 1$, then the geometric expressions on the LHS of Eq. (9) will reduce to the *product combination* of geometric expressions for starlike functions in Refs. [1, 3] and Yamaguchi functions in Ref. [21].

Example 3.3. For example, if $\alpha = 2$, then condition (9) will reduce to the analytic functions

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^2 - 1 \prec_q (m(z)-1)$$
$$\frac{f'(z)f(z)}{z} - 1 \prec_q (m(z)-1) \tag{10}$$

and if we substitute Eqs. (1) and (5) into Eq. (10), then we get:

$$3a_2z + 2(a_2^2+2a_3)z^2 + 5(a_2a_3+a_4)z^3 +$$
$$3(2a_2a_4 + a_3^2 + 3a_5)z^4 + \cdots \prec_q B_1z + B_2z^2 + B_3z^3 + B_4z^4 + \cdots .$$

Notably, functions $f \in \mathbb{Q}_q^\alpha(m)$ are associated with the class of functions with positive real parts. The objectives of this paper is to investigate the upper estimates for the bounds on $|a_r|$ $(r \in \{2,3\})$, the Fekete-Szegö functional $|a_3-\gamma a_2^2|$ and the Hankel determinant $|\Delta_{2,1}(f)|$ for functions $f \in \mathbb{Q}_q^\alpha(m)$.

Theorem 3.4. *If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then*

$$|a_2| \le \frac{|A_0|B_1}{1+\alpha} \tag{11}$$

and

$$|a_3| \le \frac{|A_1|B_1 + |A_0|B_1 + |A_0||B_2| + \frac{|\alpha-1|(\alpha+2)|A_0^2|B_1^2}{4(1+\alpha)^2}}{2+\alpha}. \tag{12}$$

Proof. Let $f \in \mathbb{Q}_q^\alpha(m)$, then there exist the analytic functions ω, ϖ and m in Eqs. (2), (4), and (5), respectively; such that Eq. (9) can be expressed as

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\alpha - 1 = \varpi(z)(m(\omega(z))-1). \tag{13}$$

Considering the RHS of Eq. (13) means that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left\{c_1z + \left[\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_1^2}{2}\right]z^2 + \left[\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + c_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_1^3}{8}\right]z^3+\cdots\right\}$$

and

$$\varpi(z)\left(m\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) = \frac{A_0B_1c_1}{2}z + \left[\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\right]z^2$$
$$+\left[\frac{1}{2}A_2B_1c_1 + \frac{1}{4}A_1B_2c_1^2 + \frac{A_0B_2}{2}c_1\left(c_2-\frac{c_1^2}{2}\right) + A_1B_1\left(c_2-\frac{c_1^2}{2}\right) + \frac{A_0B_1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)+\frac{A_0B_3}{8}c_1^3\right]z^3+\cdots . \tag{14}$$

Considering the LHS of Eq. (13) implies that

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\alpha - 1 = (1+\alpha)a_2z + \left[(2+\alpha)a_3 + \frac{(\alpha-1)(\alpha+2)}{2}a_2^2\right]z^2$$
$$+\left[(3+\alpha)a_4 + \alpha(\alpha-1) + 3(\alpha-1)a_2a_3 \times \left(\frac{\alpha(\alpha-1)(\alpha-2)}{6}+\frac{\alpha(\alpha-1)}{2}-\alpha+1\right)a_2^3\right]z^3+\cdots . \tag{15}$$

Equating Eqs. (14) and (15) yields

$$a_2 = \frac{A_0B_1c_1}{2(1+\alpha)} \implies |a_2| \le \frac{|A_0|B_1|c_1|}{2(1+\alpha)} \tag{16}$$

and the application of Lemma 2.1 in Eq. (16) gives the estimate in Eq. (11). Also note that

$$a_3 = \frac{A_1B_1c_1 + A_0B_1\left(c_2-\frac{c_1^2}{2}\right) + \frac{A_0B_2}{2}c_1^2 - \frac{(\alpha-1)(\alpha+2)A_0^2B_1^2}{8(1+\alpha)^2}c_1^2}{2(2+\alpha)} \tag{17}$$

which implies that

$$|a_3| \le \frac{|A_1|B_1|c_1| + |A_0|B_1\left|c_2-\frac{c_1^2}{2}\right| + \frac{|A_0||B_2|}{2}|c_1^2| + \frac{|(\alpha-1)|(\alpha+2)|A_0^2|B_1^2|}{8(1+\alpha)^2}|c_1^2|}{2(2+\alpha)} \tag{18}$$

and the application of Lemmas 2.1 and 2.2 in Eq. (18) gives the estimate in Eq. (12). □

Theorem 3.5. *If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then*

$$\nabla(\gamma,f) = |a_3-\gamma a_2^2| \le \begin{cases} \frac{|A_0|B_1}{2(2+\alpha)}(-\psi) + \frac{|A_1|B_1}{(2+\alpha)} & for \quad \gamma \le \tau_1 \\ \frac{|A_0|B_1}{(2+\alpha)} + \frac{|A_1|B_1}{(2+\alpha)} & for \quad \tau_1 \le \gamma \le \tau_2 \\ \frac{|A_0|B_1}{2(2+\alpha)}\psi + \frac{|A_1|B_1}{(2+\alpha)} & for \quad \gamma \ge \tau_2 \end{cases},$$

where $\gamma \in \mathbb{R}$,

$$\psi = \frac{(\alpha-1)(\alpha+2)A_0B_1^2 + 4\gamma A_0B_1^2(2+\alpha) - 4(1+\alpha)^2B_2}{2B_1(1+\alpha)^2},$$

$$\tau_1 = \left(\frac{B_2}{B_1} - \frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2} - 1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)},$$

and

$$\tau_2 = \left(\frac{B_2}{B_1} - \frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2} + 1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)}.$$

Proof. Putting Eqs. (16) and (17) into Eq. (7) means that

$$a_3 - \gamma a_2^2 = \left(\frac{A_1B_1}{2(2+\alpha)}c_1 + \frac{A_0B_1}{2(2+\alpha)}\left(c_2-\frac{c_1^2}{2}\right) + \frac{A_0B_2}{4(2+\alpha)}c_1^2 - \frac{(\alpha-1)(\alpha+2)A_0^2B_1^2}{16(1+\alpha)^2(2+\alpha)}c_1^2\right) - \gamma\left(\frac{A_0B_1}{2(1+\alpha)}c_1\right)^2$$

where some simplifications lead to

$$a_3-\gamma a_2^2=\frac{A_0B_1}{2(2+\alpha)}\left\{c_2-\left(1-\frac{B_2}{B_1}+\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}\right.\right. \\ \left.\left.+\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}\right)\frac{c_1^2}{2}\right\}+\frac{A_1B_1}{2(2+\alpha)}c_1$$

so that

$$\left|a_3-\gamma a_2^2\right|\le\frac{|A_0|B_1}{2(2+\alpha)}\left|c_2-\lambda\frac{c_1^2}{2}\right|+\frac{|A_1|B_1}{2(2+\alpha)}|c_1| \tag{19}$$

for

$$\lambda=1-\frac{B_2}{B_1}+\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}+\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}. \tag{20}$$

Using Lemma 2.2 in Eq. (19) implies that for $\lambda\le 0$,

$$\left|c_2-\lambda\frac{c_1^2}{2}\right|\le 2\left[1-\left(1-\frac{B_2}{B_1}+\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}\right.\right. \\ \left.\left.+\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}\right)\right] \\ =-\left(\frac{(\alpha-1)(\alpha+2)A_0B_1^2+4\gamma A_0B_1^2(2+\alpha)-4(1+\alpha)^2B_2}{2B_1(1+\alpha)^2}\right) \tag{21}$$

and from Eq. (20) we get

$$\gamma\le\left(\frac{B_2}{B_1}-\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}-1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)}. \tag{22}$$

Secondly, for $0\le\lambda\le 2$, we get from Eq. (19) that

$$\left|c_2-\lambda\frac{c_1^2}{2}\right|\le 2, \tag{23}$$

and from Eq. (20) we get

$$\left(\frac{B_2}{B_1}-\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}-1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)}\le\gamma\le \\ \left(\frac{B_2}{B_1}-\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}+1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)}. \tag{24}$$

Thirdly, for $\lambda\ge 2$, we get from Eq. (19) that

$$\left|c_2-\lambda\frac{c_1^2}{2}\right|\le 2\left[\left(1-\frac{B_2}{B_1}+\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}+\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}\right)\right. \\ \left.-1\right]=\frac{(\alpha-1)(\alpha+2)A_0B_1^2+4\gamma A_0B_1^2(2+\alpha)-4(1+\alpha)^2B_2}{2B_1(1+\alpha)^2}, \tag{25}$$

and from Eq. (20) we get

$$\gamma\ge\left(\frac{B_2}{B_1}-\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}+1\right)\frac{(1+\alpha)^2}{A_0B_1(2+\alpha)}. \tag{26}$$

In summary, using Lemma 2.1 and putting Eqs. (21) – (26) into Eq. (19) yield the desired result in the theorem. □

Theorem 3.6. *If $f\in\mathbb{Q}_q^{\alpha}(m)$ is given by Eq. (1), then*

$$\nabla(\gamma,f)=|a_3-\gamma a_2^2|\le\frac{|A_0|B_1}{(2+\alpha)}\max\{1,\chi\}+\frac{|A_1|B_1}{(2+\alpha)},$$

where $\gamma\in\mathbb{C}$ and

$$\chi=\left|\frac{B_2}{B_1}-\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}-\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}\right|.$$

Proof. For $\gamma\in\mathbb{C}$ in Eq. (19) and using Lemma 2.2 implies that

$$1-\lambda=1-\left(1-\frac{B_2}{B_1}+\frac{(\alpha-1)(\alpha+2)A_0B_1}{4(1+\alpha)^2}+\frac{\gamma A_0B_1(2+\alpha)}{(1+\alpha)^2}\right).$$

Some simplifications and the use of Lemma 2.1 yield the desired result. □

Theorem 3.7. *If $f\in\mathbb{Q}_q^{\alpha}(m)$ is given by Eq. (1), then*

$$|\Delta_{2,1}(f)|=|a_3-a_2^2|\le\frac{B_1(|A_0|+|A_1|)}{2+\alpha}.$$

Proof. Letting $\gamma=1$ in Theorem 3.5 yields the desired result. □

4. CONCLUSION

The investigations in this paper were on a subclass of analytic-univalent functions defined in the unit disk $\Omega:=\{z\mid z\in\mathbb{C},\ |z|<1\}$ herein denoted by $\mathbb{Q}_q^{\alpha}(m)$. This new class generalized some well-known subclasses of analytic-univalent functions which included the classes of starlike functions, Yamaguchi functions and Ma-Minda functions. The definition of the class involved the use of the principles of Taylor's series and quasi-subordination. The quasi-subordination is known to generalize the principles of subordination and majorization. The investigated properties of the function $f\in\mathbb{Q}_q^{\alpha}(m)$ however included, the estimates for some coefficient bounds, the solution to the Fekete-Szegö problem and the estimate for a Hankel determinant. Some particular fields of application include conformal mappings, special functions, physics and engineering designs.

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