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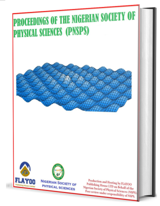

Derivative block methods for the solution for fourth-order boundary value problems of ordinary differential equations

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A B S T R A C T

The application of block methods in the study of dynamical system is one area which needs more research. Therefore, this work presents block methods and its application to solve some problems in solid and fluid mechanics. The method was developed directly using the approaches of collocation and interpolation, and using power series polynomial as a trial solution. First, the system of linear equations is solved to obtain the unknown coefficients. The coefficients gotten are then substituted into the approximate solution to obtain continuous scheme. The continuous scheme, its first, second and third derivatives are evaluated at all the grid points to generate the block methods. The derived methods were applied to solve fourth-order boundary value problems of ordinary differential equations arising from beams and chemical problems. The results demonstrate the reliability and efficiency of the proposed method.

Keywords: Derivative block methods, Stability analysis, Fourth-order differential equations, Interpolation and collocation.

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1. INTRODUCTION

Generally, fourth-order initial and boundary value problems frequently arise in fields like sciences and engineering, such as modelling viscoelastic and inelastic flows, deformation of beams, plate deflection theory and many other applications of engineering and applied mathematics. However, it is commonly known that several fourth ODEs do not have theoretical solutions.

Many researchers have developed methods for the direct numerical integration of boundary value problems (BVPs) fourth-order ordinary differential equations of the form

$$y^{iv}(t) = (t, y,\ y',\ y'',\ y'''), \tag{1}$$

with conditions

$$y(t) = \alpha_1,\ y'(t) = \alpha_2,\ y''(t) = \alpha_3,\ y'''(t) = \alpha_4. \tag{2}$$

According to Manni *et al.* [1], boundary value problems could be solved by reducing them to first-order boundary value problems with twice dimensions. Notable scholars such as Brugnano & Trigiante [2], Amodio & Sgura [3], and Asche *et al.* [4], among others, have transformed fourth-order boundary value problems into a first-order boundary value problems with doubled dimension in order to be able to get numerical solutions. However, this strategy is costly because several researchers found that converting higher-order ODEs into first-

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order ODE systems will increase the equation count. Consequently, more function evaluations must be calculated, requiring more computational effort and longer time. Many researchers have suggested a direct numerical approach to more accurate results with less calculation time.

Many scholars have proposed direct methods to solve higher-order boundary value problems of ODE's, researcher like Quang *et al.* [5] who proposed a new fixed-point approach for a fully nonlinear fourth-order problem. Their problem models the bending equilibrium of a beam on an elastic foundation whose two ends are supported, and their iterative method converges. Zhang *et al.* [6] proposed a numerical solution to the Euler-Bernoulli beam equation using the barycentric Lagrange interpolation collocation method.

Numerical approaches for the solution of fourth-order boundary value problems are numerous in the literature. Some of those methods proposed for obtaining the approximate solution of fourth-order BVPs include but are not limited to Variational Iteration Method (VIM) by Noor *et al.* [7], Quintic Spline method by Siddiqi & Akram[8], Spline-based methods by Kasi *et al.* [9], the Least Value Method by Yao & Cui, [10]. Other approaches are based on collocation methods, Variation of Parameter methods, Adomian Decomposition methods, and Differential Transform Methods, to mention a few.

According to Ramos & Rufai [11], there are mainly three different types of approximation methods for solving boundary value problems of ODEs: the shooting method, finite-difference methods, and the class of methods based on approximating the solution by a linear combination of trial functions (of which collocation methods, Galerkin method, and Rayleigh-Ritz method are the most typical examples). The shooting method transforms the boundary-value ODE into a system of First-order ODEs, which a suitable initial-value solver must solve. The finite-difference approach constructs a finite difference approximation of the exact ODE at selected points on a discrete grid, including the boundary conditions. This way, a system of coupled finite difference equations results must be solved simultaneously, thus obtaining the approximate solution at the grid points. Prominent researchers like Chen *et al.* [12], Cheng & Zhong [13], Lomtatidze *et al.* [14] and Thompson *et al.* [15] have applied finite difference methods to solve fourth-order boundary value problems together with selected boundary conditions. As seen in much literature, these methods' drawbacks are that they require significant computational costs to obtain high accuracy.

According to Dang Quang [16],there are many methods for the numerical solution of two-point nonlinear BVPs for fourth-order equations, which he said can be grouped into three. The first type includes methods for constructing discrete systems corresponding to BVPs; researchers such as Hajji & Al-Khaled [17], Mohanty [18], Siddiqi & Akram [8] and Srivastava *et al.* [19] studied the convergence of the discrete systems without any analysis of errors arising in solving the discrete systems. The second type of method is related to the methods of construction of iterative methods on the continuous level without attention to how to realize continuous problems at each iteration and errors arising at each iteration; Agarwal & Chow,[20], Azarnavid *et al.* [21], [22], Dang *et al.* [23] and Dang & Dang,[24] used this approach.

The third type includes analytical methods such as the Adomian decomposition method as found in Singh *et al.* [25], the variational iteration method by Noor *et al.* [7], the reproducing kernel method by Geng [26], when the solution is sought in series form. Spectral methods also belong to the third type since the exact solution of the problems is expressed in series representation by basis functions. The total error of the obtained approximate numerical solution has not been addressed in all the types of methods. The problem of total error in the numerical solution of nonlinear BVPs must be investigated because the total error gives helpful information for balancing discretization error and error of iterative process.

2. DERIVATION OF THE BLOCK METHOD

In this research, we consider the numerical solution of general fourth-order boundary value problems (BVPs) of ordinary differential equations of the form (1) with conditions (2).

Assume that the theoretical solution to (1) is approximated here by a polynomial of the form:

$$y(t) = p(t) = \sum_{r=0}^{k} a_r t^r, \tag{3}$$

where $a_j \in R$ are unknown coefficients to be determined and t is continuously differentiable.

The successive derivatives of (3) are obtained to be

$$\begin{aligned} y'(t) &= p'(t) \sum_{r=1}^{k} D_{1,r} a_r t^{r-1}, \\ y''(t) &= p''(t) \sum_{r=2}^{k} D_{2,r} a_r t^{r-2}, \\ y'''(t) &= p'''(t) \sum_{r=3}^{k} D_{3,r} a_r t^{r-3}, \\ y^{(4)}(t) &= p^{iv}(t) \sum_{r=4}^{k} D_{4,r} a_r t^{r-4}, \\ y^{(5)}(t) &= p^{v}(t) \sum_{r=4}^{k} j D_{5,r} a_r t^{j-5}, \end{aligned} \tag{4}$$

where

$$\prod_{s=0}^{j-1} (r-s), j = 1, 2, \ldots, 5,$$

using the approximations in (3) and (4). This leads to a system of linear equations that can be expressed in matrix form as

$$X = \begin{bmatrix} x_{11} & x_{12} & . & . & . & x_{1n} \\ x_{21} & x_{22} & . & . & . & x_{2n} \\ x_{31} & x_{32} & . & . & . & x_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{n1} & x_{n2} & . & . & . & x_{nn} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}. \tag{5}$$

We proceed to solve (5) using Gaussian elimination with the aid of CAS Mathematica to get the parameters $a_j's$, these values are subtituted back into (3) which after some simplifications gives a continuous scheme representation of the approximating polynomial in the form

$$\begin{aligned} A^0(x_n)Y_m &= A_0^{(i)}(x_n)Y_{m-i} + hA_1{}^{(i)}(x_n)Y'_{m-i} + h^2A_2{}^{(i)}(x_n)Y''_{m-i} \\ &+h^3A_3{}^{(i)}(x_n)Y'''_{m-i} + h^4\sum_{i=0}^{k}B^{(i)}(x_n)F_{m-i} \\ &+h^5\sum_{i=0}^{k}C^{(i)}(x_n)F'_{m-i}. \end{aligned} \quad (6)$$

The continuous scheme shall be evaluated to obtain the block methods. Following the above procedure as stated above from equation (3) to (6), the block methods below are derived.

2.1. THE ONE-STEP BLOCK METHOD

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n \\ &+h^4\left(\frac{f_n}{28} + \frac{f_{n+1}}{168}\right) + h^5\left(\frac{f'_n}{252} - \frac{f'_{n+1}}{630}\right). \end{aligned} \quad (7)$$

$$\begin{aligned} y'_{n+1} &= y' + hy'' + \frac{1}{2}h^2y''' + h^3\left(\frac{2f_n}{15} + \frac{f_{n+1}}{30}\right) \\ &+h^4\left(\frac{f'_n}{60} - \frac{f'_{n+1}}{120}\right). \end{aligned} \quad (8)$$

$$\begin{aligned} y''_{n+1} &= y''_n + hy''' + h^2\left(\frac{21f_n + 9f_{n+1}}{60}\right) \\ &+h^3\left(\frac{f'_n - 2f'_{n+1}}{60}\right). \end{aligned} \quad (9)$$

$$y'''_{n+1} = y'''_n + h\left(\frac{6f_n + 6f_{n+1}}{12}\right) + h^2\left(\frac{f'_n - 6f'_{n+1}}{12}\right). \quad (10)$$

2.2. THE TWO-STEP BLOCK METHOD

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''{}_n \\ &+h^4\left(\frac{67f_n}{2016} + \frac{f_{n+1}}{144} + \frac{f_{n+2}}{672}\right) \\ &+h^5\left(\frac{37f'_n}{12096} - \frac{4f'_{n+1}}{945} - \frac{5f'_{n+2}}{12096}\right). \end{aligned} \quad (11)$$

$$\begin{aligned} y_{n+2} &= y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''{}_n \\ &+h^4\left(\frac{8f_n}{21} + \frac{16f_{n+1}}{63} + \frac{2f'_{n+2}}{63}\right) \\ &+h^5\left(\frac{8f'_n}{189} - \frac{16f'_{n+1}}{189} - \frac{8f_{n+2}}{945}\right). \end{aligned} \quad (12)$$

$$\begin{aligned} y'_{n+1} &= y'_n + hy''_n + \frac{1}{2}h^2y'''{}_n \\ &+h^3\left(\frac{817f_n + 256f_{n+1} + 47f_{n+2}}{6720}\right) \\ &+h^4\left(\frac{83f'_n - 140f'_{n+1} - 13f'_{n+2}}{6720}\right). \end{aligned} \quad (13)$$

$$\begin{aligned} y'_{n+2} &= y'_n + 2hy''_n + 2h^2y'''{}_n \\ &+h^3\left(\frac{68f_n + 64f_{n+1} + 8f_{n+2}}{105}\right) \\ &+h^4\left(\frac{8f'_n - 16f'_{n+1} - 2f'_{n+2}}{105}\right). \end{aligned} \quad (14)$$

$$\begin{aligned} y''_{n+1} &= y''_n + hy'''{}_n \\ &+h^2\left(\frac{520f_n + 280f_{n+1} + 40f_{n+2}}{1680}\right) \\ &+h^3\left(\frac{59f'_n - 128f'_{n+1} - 11f'_{n+2}}{1680}\right). \end{aligned} \quad (15)$$

$$\begin{aligned} y''_{n+2} &= y''_n + 2hy'''{}_n \\ &+h^2\left(\frac{79f_n + 112f_{n+1} + 19f_{n+2}}{105}\right) \\ &+h^3\left(\frac{10f'_n - 16f'_{n+1} - 4f'_{n+2}}{10}\right). \end{aligned} \quad (16)$$

$$\begin{aligned} y'''{}_{n+1} &= y'''{}_n \\ &+h\left(\frac{101f_n + 128f_{n+1} + 11f_{n+2}}{240}\right) \\ &+h^2\left(\frac{13f'_n - 40f'_{n+1} - 3f'_{n+2}}{240}\right). \end{aligned} \quad (17)$$

$$\begin{aligned} y'''{}_{n+2} &= y'''{}_n + h\left(\frac{7f_n + 16f_{n+1} + 7f_{n+2}}{15}\right) \\ &+h^2\left(\frac{f'_n - f'_{n+2}}{15}\right). \end{aligned} \quad (18)$$

3. ANALYSIS OF THE METHOD

3.1. LOCAL TRUNCATION ERROR AND ORDER

Following the procedure stated by Fatunla [27] and Lambert [28], it is possible to show that the block methods derived are of $2k+2$. Thus,we have the following Lemma.

3.1.1. Lemma

1. The order of the block method for $k = 1, 2, 3\cdots$ is $2k + 2$.

3.1.2. Proof

The general form of the block methods is

$$\begin{aligned} y(x) = \sum_{i=0}^{3}\alpha_i y_n^i(x_n)h^i + h^4\sum_{i=0}^{k}\beta_i(x_n)f_{n+i} \\ + h^5\sum_{i=0}^{k}\beta_i(x_n)f'_{n+i}, \end{aligned} \quad (19)$$

assuming,

$$\begin{aligned} y_{n+v} &\approx y(x_n + vh), f_{n+j} \equiv (x_n + jh, y(t_n + jh)), \\ f'_{n+v} &= \frac{df(t, y(t))}{dt}\Big|_{y=y_{n+v}}^{x=x_{n+v}} \end{aligned}$$

and $y(x_n)$ is an arbitrary function continuously differentiable on $[a, b]$.

We define the local truncation error (LTE) associated with the

Table 1. Comparison of methods for problem 1.

h	*Exact*	*E3SBM*	Modebei *et al.* [30]
0.1	0.01981000000000000	3.469446951953614E-18	0
0.2	0.07712000000000000	1.3877787807814457E-17	1.39E-17
0.3	0.16623000000000000	0	2.78E-17
0.4	0.27904000000000000	5.551115123125783E-17	0
0.5	0.40625000000000000	5.551115123125783E-17	0
0.6	0.53856000000000000	0	0
0.7	0.66787000000000000	1.1102230246251565E-16	0
0.8	0.78848000000000000	2.220446049250313E-16	1.11E-16
0.9	0.89829000000000000	1.1102230246251565E-16	1.11E-16
1.0	1.00000000000000000	0	0

family of the block as the linear operator $L[y(x); h]$ such that

$$L[y(x);h] = y(x_n + ih) - \Big(\alpha_1 hy'(x) - \alpha_2 h^2 y''(x) - \alpha_3 h^3 y'''(x) - h^4 \sum_{j=0}^{k} \beta_j(x) f_{n+j} - h^5 \sum_{j=0}^{k} \gamma_j f'_{n+j}\Big). \quad (20)$$

Expanding the right hand side (RHS) in Taylor series about point t, the order of the method is $(k+1)+(k+1)$, i.e., $p = (k+1)+(k+1) = 2k+2$.

The Local Truncation Error (L.T.E) is

$$\begin{aligned} L.T.E.(x_n) &= -y(x) + y_n + \alpha_1 hy'(x) + \alpha_2 h^2 y''(x) \\ &\quad + \alpha_3 h^3 y'''(x) + h^4 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \\ &\quad + h^5 \sum_{j=0}^{k} \gamma_j f'_{n+j} \\ &\leq C_{p+1} h^{p+1} y_n^{p+1}(x) + O\left(h^{(p+1)}\right) \\ &= C_{2k+3} h^{2k+3} y_n^{2k+3}(x) + O\left(h^{(2k+4)}\right). \end{aligned} \quad (21)$$

The order for $K = 1, 2, 3$, are $[4,4,4,4]^T$, $[6,6,6,6,6,6,6,6]^T$ and $[8,8,8,8,8,8,8,8,8,8,8,8]^T$, respectively.

Local truncation errors are

$$C_{p+4} = \left[\frac{1}{24192}, \frac{1}{5040}, \frac{1}{1440}, \frac{1}{720}\right]^T,$$

$$C_{p+4} = \left[\frac{1}{259200}, \frac{1}{14175}, \frac{1}{56700}, \frac{2}{14175}, \frac{1}{17280}, \frac{1}{4725}, \frac{1}{9450}, \frac{1}{4725}\right]^T,$$

and

$$C_{p+4} = \left[\frac{3359}{6706022400}, \frac{221}{26195400}, \frac{1053}{27596800}, \frac{89}{39916800}, \frac{1}{62370}, \frac{81}{1724800}, \frac{359}{50803200}, \frac{17}{793800}, \frac{27}{627200}, \frac{313}{25401600}, \frac{13}{793800}, \frac{9}{313600}\right]^T,$$

respectively.

Table 2. Comparison of methods for problem 2.

h	*Exact*	*E2SBM*	Dang Quang [32]
30	1.0833333333333333	1.1102230246251565E-15	0.0065
50	1.0833333333333333	2.6645352591003757E-15	0.0021
100	1.0833333333333333	2.4424906541753444E-15	3.9522E-04
200	1.0833333333333333	4.440892098500626E-16	3.9522E-04
500	1.0833333333333333	9.992007221626409E-15	3.9522E-04

Table 3. Comparison of methods for problem 3.

h	$y - exact$	E3SBM	Jator [31]
0	0	0	0
0.25	0.009122721354166667	8.153200337090993E-17	3.46945E-17
0.5	0.031510416666666666	1.3877787807814457E-17	2.28983E -16
0.75	0.0605712890625	1.3183898417423734E-16	3.67761E-16
1.0	0.09166666666666666	2.0816681711721685E-16	5.13478E-16

3.2. ZERO STABILITY OF THE METHODS

The general form of block method is given as :

$$A^{(0)} Y_m = A^{(r)} Y_{m-1} + h^{\mu}[B^{(i)} F_m + B^{(0)} F_{m-1}. \quad (22)$$

A method is said to be zero stable, if the roots of

$$det[\lambda A^{(0)} - A^{(r)}] = 0. \quad (23)$$

First characteristic polynomial satisfies $|\lambda| \leq 1$ and for the roots with $|\lambda| = 1$, the multiplicity must not exceed the order of the differential equations according to Fatunla [29].

This kind of stability issue concerned with the behaviour of the different system when h→ 0. For h→ 0, the system of equations can be written as:

$$A^{(0)} Y_m = A^{(r)} Y_{m-1}, \quad (24)$$

where $A^{(0)}$ is identity matrix.

The roots of the methods for $k = 1, 2, 3$ are as follows
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 1$
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0, \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 1$,
respectively.

4. NUMERICAL APPLICATION

In this section, we have tested the performance of our method on some initial and boundary value problems including linear, non-linear and system of equation. For each example, we find the absolute errors of the approximate solution and compare them with various existing methods in the literature. We note that the accuracy of our method is measured by the small error values obtained.

ACRONYMS

1SBM - One-Step Block Method
2SBM - Two-Step Block Method
3SBM - Three-Step Block Method
E1SBM - Error in One-Step Block Method
E2SBM - Error in Two-Step Block Method
E3SBM - Error in Three-Step Block Method

Table 4. Comparison of methods for problem 4.

h	*E1SBM*	*E2SBM*	*E3SBM*
0.31415	5.225942104137085E-07	3.-10	4.597767479241899E-13
0.62831	1.0739644691806077E-06	3.823533729213624E-10	9.93960122541715E-13
0.94247	1.0423257126336571E-06	9.285122150315406E-11	1.0277282497250795E-12
1.25663	5.933619030536533E-07	6.617654341178891E-11	6.183465198206228E-13
1.57079	1.1284451988122402E-18	2.7394099603923085E-19	8.490722256154154E-19
1.88495	5.933619030545206E-07	6.617654167706544E-11	6.183439177354089E-13
2.19911	1.0423257126319224E-06	9.285122497260101E-11	1.0277299844485555E-13
2.51327	1.0739644691806077E-06	3.823533729213624E-10	9.93956653094763E-13
2.82743	5.225942104119738E-07	3.5321875892918575E-10	4.597763142433209E-13
3.14159	0	0	0

Problem 1

Consider the nonlinear boundary value problem

$$y^{iv} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

$$0 < x < 1,\ y(0) = 0,\ y(1) = 1;\ y'(0) = 0,\ y'(1) = 1,\ h = 0.1$$

Exact solution is

$$y(x) = x^5 - 2x^4 + 2x^2.$$

Problem 2

$$y'^{v} = -18 + \frac{1}{5}y^2 - \frac{1}{5}\left(\frac{5}{6} + x^2 + \frac{3}{4}x^4\right)^2, \quad 0 < x < 1$$

$$y(0) = \frac{5}{6}, y'(0) = 0, y'(1) = 0; y''(0) = 0; h = 0.1$$

Exact solution is

$$y(x) = \frac{5}{6} - x^3 + \frac{3}{4}x^4.$$

Table 5. Comparison of methods for problem 5.

h	E3SBM	Ullah *et.al*. [33]	Adeyeye & Omar[34]
0.1	5.551115123125783E-17	7.58785506649317E-10	4.032885E-14
0.2	2.220446049250313E-16	1.39478356642186E-09	2.363387E-13
0.3	3.3306690738754696E-16	1.80948145356296E-09	6.848411E-13
0.4	3.3306690738754696E-16	1.94815263920844E-09	1.489975E-12
0.5	0	1.81075676675135E-09	2.768896E-12
0.6	0	1.45204859247627E-09	4.502620E-12
0.7	2.220446049250313E-16	9.70818092582703E-10	6.029954E-12
0.8	3.3306690738754696E-16	4.90335771985428E-10	6.408873E-12
0.9	7.771561172376096E-16	1.33847599670389E-10	4.708123E-12
1.0	0	2.22044604925031E-16	0

Application to dynamical problems

Problem 3

Considering the given linear BVP that involves a cantilever beam of length L with both ends fixed, distributed load, $k(x)$, modulus of elasticity E and moment of inertial I. The problem is solved for $k(x) = x, L = 1$, and $EI = 1$

$$EI\frac{d^4y}{dx^4}(x) = K(x),\ y(0) = 0,\ y'(0) = 0,\ y''(L) = 0\ y'''(L) = 0.$$

Exact solution is

$$y(x) = \frac{1}{120}\left(20x^2 - 10x^3 + x^5\right).$$

Problem 4:

Consider the problem of bending a rectangular clamped beam of length π resting on an elastic foundation. The vertical deflection ω of the beam satisfies the system:

$$\frac{d^4\omega}{dx^4} + 64\omega = \sin(2x), \qquad 0 < x < \pi,$$

$\omega(0) = \omega'(0) = \omega(\pi) = \omega'(\pi) = 0; h = 0.1.$
Exact solution is

$$\omega(x) = \frac{-(-1 + e^{2x})(-e^{2x} + e^{2x})\sin(2x)}{80e^{2x}(1 + e^{2\pi})}.$$

Problem 5

Investigating magnetohydrodynamics (MHD) squeezing flow of Newtonian fluid between two parallel plates passing through porous medium, the governing partial differential equations after some simplification reduce to single PDE as follows:

$$\rho\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{\eta^2\psi}{r^2}\right) - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{\eta^2\psi}{r^2}\right)\right] = -\frac{\mu}{r}\eta^4\psi + \frac{\mu}{k}\eta^2\psi + \frac{\eta {B_0}^2}{r}\frac{\partial^2\psi}{\partial z^2},$$

Here, $\eta = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\left(\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$. If the moving plates are separated by distance $2d$, then
$u_r = 0, \quad u_z = -v, \quad$ at $\quad z = d, \quad u_z = 0$
$\frac{\partial u_r}{\partial z} = 0$ at $z = 0$.
Using the transformation $\psi(r, z) = r^2 f(z)$, it reduces the PDE to an ODE of the form:

$$\frac{d^4}{dz^4}f(z) + \frac{2\rho}{\mu}f(z)\frac{d^3}{dz^3}f(z) - \frac{1}{k}\frac{d^2}{dz^2}f(z) - \frac{\sigma {B_0}^2}{\mu}\frac{d^2}{dz^2}f(z) = 0,$$

with the conditions
$\frac{d^2}{dz^2}f(0) = 0, \quad f(0) = 0, \quad \frac{d}{dz}f(d) = 0, \quad f(d) = -\frac{u}{2}.$
Using non-dimensional parameters

$$\left(\frac{u}{2}\right)f^* = f,\ dz^* = z,\ \left(\frac{u}{\mu}\right)R_{mp} = \rho d,\ m_h = \frac{\sigma d B_0}{\mu},$$

and $m_p = \frac{h}{k}$ and omitting the $*$, we have

$$\frac{d^4}{dz^4}f(z) + R_{mp}f(z)\frac{d^3}{dz^3}f(z) - m_p\frac{d^2}{dz^2}f(z) - m_h\frac{d^2}{dz^2}f(z) = 0,$$

$$\frac{d^2}{dz^2}f(0) = 0,\ f(0) = 0,\quad \frac{d}{dz}f(1) = 0,\ f(1) = 0,$$

where R_{mp} is Reynold and m_h, m_p are Hartmann numbers.

5. CONCLUSION

A family of block methods of $(2k+2)$ order has been derived and analysed in this work for the direct approximation of fourth-order boundary value problems in ordinary differential equations in dynamical systems. The method is simple in terms of derivation and implementation to solve a variety of boundary value problems with different boundary conditions. The theoretical analysis of the methods shows that it is convergence and its application to some numerical examples established the efficiency and high accuracy of the proposed family of method which make it competitive with other methods in the literature.

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