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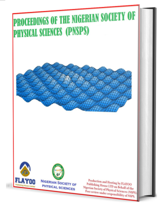

Exploring fixed point results for condensed Kannan-type cyclic maps

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ABSTRACT

Condensed Kannan-type contraction has recently been introduced to investigate the properties of nonlinear operators in classical metric spaces. However, certain classes of nonlinear operators cannot be adequately analyzed within this standard framework because of its restrictive geometric structure. To address this limitation, this paper introduces a new class of mappings, namely qp_b-cyclic condensed Kannan-type contraction mappings, in the setting of quasi-partial b-metric spaces. The main aim is to employ this novel approach to establish fixed point existence and uniqueness results under qp_b-condensed Kannan-type contractive conditions. By integrating cyclic mapping techniques with the generalized geometry of quasi-partial b-metric spaces, the proposed framework extends and unifies several existing results in the literature. The effectiveness and applicability of the obtained results are further illustrated with appropriate examples.

Keywords: Condensed Kannan-type map, Unique fixed point, Non-unique fixed point, Quasi-partial b-metric space, Cyclic mapping.

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1. INTRODUCTION

Since Banach's contribution [1] to contraction mapping in classical metric spaces, the underlying metric framework has undergone extensive generalization to accommodate richer geometric structures and broader applications. In particular, the standard triangle inequality, while fundamental, can be restrictive in certain analytical settings. To address this limitation, alternative structures such as b-metric spaces and partial metric spaces were introduced by Czerwik [2] and Matthews [3], respectively. These frameworks allow more flexible distance functions and have proven useful in extending fixed point theory to more generalized contexts. Subsequently, numerous extensions and refinements have been developed to investigate the existence and uniqueness of fixed points for increasingly robust classes of operators. For example, Chi *et al.* [4] examined classical contraction mappings within partial metric spaces, although their analysis did not fully address convergence rates. Later, Karapınar *et al.* [5] established fixed point results in quasi-partial metric spaces, demonstrating improved convergence behavior compared to the standard partial metric setting. This line of development was further extended to partial b-metric spaces [6], thereby combining the advantages of both generalizations. For several other notable contributions that have broadened both the theoretical foundations and applicability of generalized metric structures, see [7–10]. Parallel to the evolution of metric frameworks, significant attention has also been given to the generalization of contractive

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conditions. A notable breakthrough in this direction was made by Kannan [11], who introduced a class of non-continuous contractive mappings that still guarantee the existence of a unique fixed point in standard metric spaces. This concept has since been extended in various directions, including investigations into the interpolative Kannan-type map for approximating non-unique fixed point problems [12] and other generalized contractive maps [13–16]. In particular, recent works [17–19] have introduced condensed Kannan-type mappings, which impose dual bounding conditions to ensure existence of fixed points, thereby refining and strengthening classical Kannan-type results.

Another important development in fixed point theory is the notion of cyclical contractions, introduced by Kirk [20]. This approach considers mappings defined on unions of subsets and has proven effective in establishing fixed point results under cyclic conditions. The versatility and applicability of this concept have been demonstrated through various examples and further generalizations in recent studies such as [21–23].

Motivated by these developments [6, 16, 20, 23], the present paper introduces a novel framework that integrates condensed Kannan-type contractions with cyclic mappings in the setting of quasi-partial b-metric spaces. The novelty of this work lies in the simultaneous incorporation of: (i) a more flexible geometric structure, (ii) a strengthened contractive condition, and (iii) a cyclic mapping scheme.

This combined approach generalizes several existing results in the literature but also provides improved conditions for the existence of fixed points and potentially enhances convergence behavior. Consequently, the results presented herein contribute a unified and more robust framework for analyzing nonlinear operators in generalized metric spaces. The following definitions and results are useful in this paper.

Definition 1.1 ([6])**.** Let M be a nonempty set. A quasi-partial b-metric is a function $qp_b : M\times M \to \mathbb{R}^+$ such that for $s \geq 1$ and for all $\rho, \varrho, \delta \in M$:

- qp_b1: $qp_b(\rho,\rho) = qp_b(\rho,\varrho) = qp_b(\varrho,\varrho) \Leftrightarrow \rho = \varrho$,
- qp_b2: $qp_b(\rho,\rho) \leq qp_b(\rho,\varrho)$,
- qp_b3: $qp_b(\rho,\rho) \leq qp_b(\varrho,\rho)$,
- qp_b4: $qp_b(\rho,\varrho) \leq s[qp_b(\rho,\delta) + qp_b(\delta,\varrho)] - qp_b(\delta,\delta)$.

A quasi-partial b-metric space is a pair (M, qp_b) such that M is a nonempty set and qp_b is a quasi-partial b-metric on M.
If qp_b is a quasi-partial b-metric on M, then $qp_b(\rho,\varrho) = qp_b(\rho,\varrho) + qp_b(\varrho,\rho) - qp_b(\rho,\rho) - qp_b(\varrho,\varrho)$ is a b-metric on M. Also, the space (M, qp_b) is a quasi-partial metric on M if $s = 1$.

Example 1.2. Let $M = \mathbb{R}$. Define the metric $qp_b(\rho,\varrho) = |\rho-\varrho| + |\rho| + |\rho-\varrho|^2$ for any $(\rho,\varrho) \in M\times M$ with $s \geq 2$, then (M, qp_b) is a quasi-partial b-metric space.

qp_b1: If $qp_b(\rho,\rho) = qp_b(\varrho,\varrho) = qp_b(\rho,\varrho) \Rightarrow \rho = \varrho$ is true.
qp_b2: Obviously, $qp_b(\rho,\rho) \leq qp_b(\rho,\varrho)$.
qp_b3: Also, $qp_b(\rho,\rho) = |\rho| \leq |\rho-\varrho| + |\varrho| + |\rho-\varrho|^2$, since

$$|\rho| - |\varrho| \leq ||\rho| - |\varrho|| \leq |\rho-\varrho| \leq |\rho-\varrho| + |\rho-\varrho|^2.$$

qp_b4: For $s = 2$, we prove that:

$$qp_b(\rho,\varrho) \leq 2[qp_b(\rho,\delta) + qp_b(\delta,\varrho)] - qp_b(\delta,\delta).$$

That is, since

$$|\rho-\varrho|^2 \leq (|\rho-\delta| + |\delta-\varrho|)^2 \leq 2(|\rho-\delta|^2 + |\delta-\varrho|^2),$$

we have

$$\begin{aligned} qp_b(\rho,\varrho) + qp_b(\delta,\delta) &= |\rho-\varrho| + |\rho| + |\rho-\varrho|^2 + |\delta| \\ &\leq 2(|\rho-\delta| + |\delta-\varrho| + |\rho| + |\delta| \\ &\quad + |\rho-\delta|^2 + |\delta-\varrho|^2). \end{aligned}$$

Hence, (X, qp_b) is a quasi-partial b-metric space with $s = 2$.

Definition 1.3 ([9])**.** Let (M, qp_b) be a quasi-partial b-metric. Then

(i) A sequence $\{\rho_n\} \subset M$ converges to $\rho \in M$ if and only if

$$qp_b(\rho,\rho) = \lim_{n\to\infty} qp_b(\rho,\rho_n) = \lim_{n\to\infty} qp_b(\rho_n,\rho).$$

(ii) A sequence $\{\rho_n\} \subset M$ is called a Cauchy sequence if and only if

$$\lim_{n,m\to\infty} qp_b(\rho_n,\rho_m) \text{ and } \lim_{m,n\to\infty} qp_b(\rho_m,\rho_n) \text{ exists.}$$

.

(iii) The space (M, qp_b) is said to be complete if every Cauchy sequence $\{\rho_n\} \subset M$ converges to a point $\rho \in M$ such that

$$\begin{aligned} qp_b(\rho,\rho) &= \lim_{n,m\to\infty} qp_b(\rho_n,\rho_m) \\ &= \lim_{m,n\to\infty} qp_b(\rho_m,\rho_n). \end{aligned}$$

Definition 1.4 ([20])**.** Let (M, d) be a metric space and $C, D \subset M$. A mapping $\mathfrak{T} : C\cup D \to C\cup D$ is said to be a cyclic mapping provided

$$\mathfrak{T}(C) \subseteq D, \mathfrak{T}(D) \subseteq C.$$

The following is a fixed point theorem for the cyclic contraction map:

Theorem 1.5 ([20])**.** *Let (M, d) be a complete metric space and let $C, D \subset M$. Suppose that $\mathfrak{T} : C \cup D \to C \cup D$ is a cyclic contraction and there exists $k \in (0, 1)$ such that $d(\mathfrak{T}u, \mathfrak{T}v) \leq kd(u, v)$ for all $u \in C$ and $v \in D$. Then, $\mathfrak{T}$ possesses a unique fixed point in $C \cap D$.*

2. MAIN RESULTS

The main results of this paper are presented in two parts, namely, condensed Kannan-type cyclic and trivially cyclic maps in quasi-partial b-metric spaces.

2.1. CONDENSED KANNAN-TYPE CYCLIC CONTRACTION

We begin with the following definition of condensed Kannan-type cyclic contraction:

Definition 2.1. Let (M, qp_b, s) be a complete quasi-partial b-metric space and let $C, D \subset M$. A cyclic map $\mathfrak{T} : C \cup D \to C \cup D$ is a condensed Kannan-type cyclic contraction if there exist $\mu \in [0, \frac{1}{s})$, $\lambda \in [0, \frac{1}{2s})$, and $\alpha \in (0, 1)$ such that

$$\mu L(\rho, \varrho, \alpha) \leq qp_b(\mathfrak{T}\rho, \mathfrak{T}\varrho) \leq \lambda U(\rho, \varrho, \alpha), \tag{1}$$

where $L(\rho, \varrho, \alpha) = qp_b(\mathfrak{T}\rho, \rho)^{\alpha} qp_b(\mathfrak{T}\varrho, \varrho)^{1-\alpha}$, $U(\rho, \varrho, \alpha) = qp_b(\mathfrak{T}\rho, \rho)^{2\alpha} + qp_b(\mathfrak{T}\varrho, \varrho)^{2(1-\alpha)}$, for all $\rho, \varrho \in M \backslash Fix(\mathfrak{T})$.

The inequality (1) is a blend of the interpolative Kannan-type [12] and a modified Kannan contraction [17] in quasi-partial b-metric spaces. The image distance $qp_b(\mathfrak{T}\rho, \mathfrak{T}\varrho)$ is controlled by a bounded interval $[\mu L, \lambda U]$, where μ and λ are contracting constants; the lower bound μL controls how small the image distance can be while λU controls controls how large it can be. If the distance $qp_b(\mathfrak{T}\rho, \mathfrak{T}\varrho) \in [\mu L, \lambda U]$ for some $\alpha \in (0, 1)$, then we assert that the operator $\mathfrak{T}$ satisfies the inequality (1) for all $\rho, \varrho \in M \backslash Fix(\mathfrak{T})$.

In what follows, we deduce some consequences of inequality (1) that are evident in the literature:

I. By replacing $M \backslash Fix(\mathfrak{T})$ with M, inequality (1) becomes

$$qp_b(\mathfrak{T}\rho, \mathfrak{T}\varrho) \leq \lambda[qp_b(\mathfrak{T}\rho, \rho)^{2\alpha} + qp_b(\mathfrak{T}\varrho, \varrho)^{2(1-\alpha)}]. \tag{2}$$

That is, for some $\rho, \varrho \in M$, the interpolative term $L(\rho, \varrho, \alpha)$ vanishes.

II. As a quick note in I, we recover the Kannan contraction [11] by setting $\alpha = \frac{1}{2}$ in (2).

III. By letting $s = 1$, inequality (8) is restricted to quasi-partial metric spaces.

The existence theorem of an operator satisfying (1) is presented as follows:

Theorem 2.2. *Let $C, D \subset M$ and $\mathfrak{T} : C \cup D \to C \cup D$ is a condensed Kannan-type cyclic contraction in $M \backslash Fix(\mathfrak{T})$, then $\mathfrak{T}$ has a unique fixed point in $C \cap D$.*

Proof. Let $\mathfrak{T} : C \cup D \to C \cup D$ be a condensed Kannan-type cyclic contraction. Then, we define a sequence $\{\rho_n\}$ in $C \cup D$ such that

$$\rho_{n+1} = \mathfrak{T}\rho_n \quad \forall \quad n \in \mathbb{N} \cup \{0\}. \tag{3}$$

Fix $\rho_0 \in C$, then $\mathfrak{T}\rho_0 \in D$, $\mathfrak{T}^2\rho_0 \in C$, $\mathfrak{T}^3\rho_0 \in D$, and so on. From (3), we can write

$$\rho_1 = \mathfrak{T}\rho_0, \quad \rho_2 = \mathfrak{T}^2\rho_0,$$
$$\rho_3 = \mathfrak{T}^3\rho_0, \quad \ldots, \quad \rho_n = \mathfrak{T}^n\rho_0.$$

By adopting the $U(\rho, \varrho, \alpha)$ and $L(\rho, \varrho, \alpha)$ of inequality (1) in $M \backslash Fix(\mathfrak{T})$, we have

$$\begin{aligned} qp_b(\rho_2, \rho_1) &= qp_b(\mathfrak{T}^2\rho_0, \mathfrak{T}\rho_0) \\ &\leq \lambda[qp_b(\mathfrak{T}^2\rho_0, \mathfrak{T}\rho_0)^{2\alpha} \\ &\quad + qp_b(\mathfrak{T}\rho_0, \rho_0)^{2(1-\alpha)}] \\ &= \lambda[qp_b(\rho_2, \rho_1)^{2\alpha} + qp_b(\rho_1, \rho_0)^{2(1-\alpha)}], \end{aligned}$$

and

$$\begin{aligned} qp_b(\rho_2, \rho_1) &= qp_b(\mathfrak{T}^2\rho_0, \mathfrak{T}\rho_0) \\ &\leq \mu qp_b(\mathfrak{T}^2\rho_0, \mathfrak{T}\rho_0)^{\alpha} qp_b(\mathfrak{T}\rho_0, \rho_0)^{1-\alpha} \\ &= \mu qp_b(\rho_2, \rho_1)^{\alpha} qp_b(\rho_1, \rho_0)^{1-\alpha}, \end{aligned}$$

respectively. Continue in this manner up to the nth term, we have

$$qp_b(\rho_{n+1}, \rho_n) \leq \lambda[qp_b(\rho_{n+1}, \rho_n)^{2\alpha} + qp_b(\rho_n, \rho_{n-1})^{2(1-\alpha)}], \tag{4}$$

and

$$qp_b(\rho_{n+1}, \rho_n) \leq \mu qp_b(\rho_{n+1}, \rho_n)^{\alpha} qp_b(\rho_n, \rho_{n-1})^{1-\alpha}. \tag{5}$$

By combining (4) and (5), we obtain

$$[qp_b(\rho_{n+1}, \rho_n)^{\alpha} - qp_b(\rho_n, \rho_{n-1})^{1-\alpha}]^2 \geq 0, \tag{6}$$

which gives the following two cases:

(a) $qp_b(\rho_{n+1}, \rho_n)^{\alpha} \leq qp_b(\rho_n, \rho_{n-1})^{1-\alpha}$.

(b) $qp_b(\rho_{n+1}, \rho_n)^{\alpha} \geq qp_b(\rho_n, \rho_{n-1})^{1-\alpha}$.

Both cases (a) and (b) satisfy the equality:

$$qp_b(\rho_{n+1}, \rho_n)^{\alpha} = qp_b(\rho_n, \rho_{n-1})^{1-\alpha}.$$

Using this equality in (4), we get

$$\begin{aligned} qp_b(\rho_{n+1}, \rho_n) &\leq \mu qp_b(\rho_n, \rho_{n-1})^{2\alpha} \\ &= qp_b(\rho_n, \rho_{n-1})^{\alpha} qp_b(\rho_{n+1}, \rho_n)^{1-\alpha}. \end{aligned}$$

which further implies that

$$qp_b(\rho_{n+1}, \rho_n)^{\alpha} \leq \mu qp_b(\rho_n, \rho_{n-1})^{\alpha}$$

By induction, we deduce the following:

$$qp_b(\rho_{n+1}, \rho_n) \leq \kappa^n qp_b(\rho_1, \rho_0), \quad \text{where} \quad \kappa \equiv \mu^{\frac{1}{\alpha}}.$$

Let $m > n$ for $m \in \mathbb{N}$ and using (qp_b4), we have that

$$\begin{aligned} qp_b(\rho_n, \rho_m) &\leq s\left[qp_b(\rho_n, \rho_{n+1}) + qp_b(\rho_{n+1}, \rho_m)\right] \\ &\quad - qp_b(\rho_{n+1}, \rho_{n+1}) \\ &\leq s\Big[qp_b(\rho_n, \rho_{n+1}) + s(qp_b(\rho_{n+1}, \rho_{n+2}) \\ &\quad + qp_b(\rho_{n+2}, \rho_m)) - qp_b(\rho_{n+2}, \rho_{n+2})\Big] \\ &\quad - qp_b(\rho_{n+1}, \rho_{n+1}) \\ &\cdots \\ &\leq \sum_{i=1}^{m-n} s^i qp_b(\rho_{n+i-1}, \rho_{n+i}) \\ &\quad - \sum_{i=1}^{m-n} s^{i-1} qp_b(\rho_{n+i}, \rho_{n+i}) \\ &\leq \sum_{i=1}^{m-n} s^i \kappa^{n+i-1} \\ &\quad \times qp_b(\rho_0, \rho_1). \end{aligned}$$

Taking limit as $n \to \infty$ of the last inequality, we have

$$\lim_{n\to\infty} qp_b(\rho_n, \rho_m) = 0.$$

This proves that $\{\rho_n\}$ is a b-Cauchy sequence, by completeness, $\{\rho_n\}$ converges to $\rho^* \in M$. That is,

$$\lim_{n\to\infty} \rho_n = \rho^*.$$

Next is to show that $\rho^* = \mathfrak{T}\rho^*$. On the contrary, assume that $\rho^* \neq \mathfrak{T}\rho^*$, then

$$\begin{aligned} qp_b(\mathfrak{T}\rho^*, \rho^*) &\leq s[qp_b(\mathfrak{T}\rho^*, \mathfrak{T}\rho_n) + qp_b(\rho_{n+1}, \rho^*)] \\ &\quad - qp_b(\rho_{n+1}, \rho_{n+1}). \end{aligned} \tag{7}$$

But

$$\begin{aligned} qp_b(\mathfrak{T}\rho^*, \mathfrak{T}\rho_n) &\leq \lambda\left[qp_b(\rho^*, \mathfrak{T}\rho^*)^{2\alpha} + qp_b(\rho_n, \mathfrak{T}\rho_n)^{2(1-\alpha)}\right] \\ &= \lambda\left[qp_b(\rho^*, \mathfrak{T}\rho^*)^{2\alpha} + qp_b(\rho_n, \rho_{n+1})^{2(1-\alpha)}\right]. \end{aligned}$$

By taking limit as $n \to \infty$ across (7), this gives

$$qp_b(\mathfrak{T}\rho^*, \rho^*)^{1-2\alpha} \leq s\lambda < 1, \text{ for } \alpha \in (0, 1),$$

which is a contradiction. Hence, $\rho^* = \mathfrak{T}\rho^*$. Suppose that ϱ^* is any other fixed point of $\mathfrak{T}$ such that $qp_b(\rho^*, \varrho^*) \neq 0 \Leftrightarrow \rho^* \neq \varrho^*$. Then,

$$\begin{aligned} qp_b(\rho^*, \varrho^*) &= qp_b(\mathfrak{T}\rho^*, \mathfrak{T}\varrho^*) \\ &\leq \lambda[qp_b(\rho^*, \mathfrak{T}\rho^*)^{2\alpha} + qp_b(\varrho^*, \mathfrak{T}\varrho^*)^{2(1-\alpha)}] \\ &= \lambda[qp_b(\rho^*, \rho^*)^{2\alpha} + qp_b(\varrho^*, \varrho^*)^{2(1-\alpha)}] = 0. \end{aligned}$$

This leads to a contradiction. Hence, $\rho^* (= \varrho^*)$ is unique. □

Corollary 2.3. *Let $C, D \subset M$ and $\mathfrak{T} : C \cup D \to C \cup D$ is a condensed Kannan-type cyclic contraction in M, then $\mathfrak{T}$ has a unique fixed point in $C \cap D$.*

Proof. If we replace $M \backslash Fix(\mathfrak{T})$ with M, then the condition $L(\rho, \varrho, \alpha)$ collapses and the proof follows by using only the $U(\rho, \varrho, \alpha)$. □

Corollary 2.4. *Let $C, D \subset M$, M a quasi-partial metric space and $\mathfrak{T} : C \cup D \to C \cup D$ is a condensed Kannan-type cyclic contraction in M, then $\mathfrak{T}$ has a unique fixed point in $C \cap D$.*

Proof. The proof follows from the preceding theorem if $s = 1$. □

2.2. CONDENSED KANNAN-TYPE TRIVIALLY CYCLIC CONTRACTION

The following is a definition of trivially cyclic condensed map in quasi-partial b-metric spaces:

Definition 2.5. Let M be a quasi-partial b-metric space. A map $T : M \to M$ is called a condensed Kannan-type of trivially cyclic contraction if there exist constants $\alpha \in (0, 1)$, $\lambda \in [0, \frac{1}{2s})$, and $\mu \in [0, \frac{1}{s})$ such that

$$\begin{aligned} qp_b(T\rho, T\varrho) &\leq \lambda[qp_b(\rho, T\rho)^{2\alpha} \\ &\quad + qp_b(\varrho, T\varrho)^{2(1-\alpha)}], \quad (U(\rho, \varrho, \alpha)), \\ qp_b(T\rho, T\varrho) &\geq \mu qp_b(\rho, T\rho)^{\alpha} qp_b(\varrho, T\varrho)^{1-\alpha}, \\ &\quad (L(\rho, \varrho, \alpha)). \end{aligned} \tag{8}$$

for all $\rho, \varrho \in M \backslash Fix(T)$.

The condition (8) is valid if the fractional powers α and $1 - \alpha$ are interchanged.

The second main result is presented as follows:

Theorem 2.6. *Let $T : M \to M$ be Kannan-type contraction map (8), where M is a quasi-partial b-metric space. Then, T admits a unique fixed point in M.*

Proof. Let $\rho_0 \in M$ be fixed and let $\{\rho_n\} \subset M$ be defined by $\rho_{n+1} = T\rho_n$ for all $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}_0$ such that $\rho_{n_0+1} = T\rho_{n_0} = \rho_{n_0}$, then the proof is immediate. Now, let $\rho_{n+1} \neq \rho_n$ for each $n \in \mathbb{N}_0$; using the $(U(\rho, \varrho, \alpha))$ of (8), we get

$$\begin{aligned} qp_b(\rho_n, \rho_{n+1}) &= qp_b\left(T\rho_{n-1}, T\rho_n\right) \\ &\leq \lambda[qp_b(\rho_{n-1}, T\rho_{n-1})^{2\alpha} \\ &\quad + qp_b(\rho_n, T\rho_n)^{2(1-\alpha)}] \\ &= \lambda[qp_b(\rho_{n-1}, \rho_n)^{2\alpha} \\ &\quad + qp_b(\rho_n, \rho_{n+1})^{2(1-\alpha)}]. \end{aligned} \tag{9}$$

By using the $(L(\rho, \varrho, \alpha))$ and (9), this gives

$$\begin{aligned} 0 &\leq qp_b(\rho_n, \rho_{n+1}) - \mu qp_b(\rho_{n-1}, \rho_n)^{\alpha} qp_b(\rho_n, \rho_{n+1})^{1-\alpha} \\ &\leq \lambda\left[qp_b(\rho_{n-1}, \rho_n)^{\alpha} - qp_b(\rho_n, \rho_{n+1})^{1-\alpha}\right]^2 \text{ with } \mu \equiv 2\lambda. \end{aligned} \tag{10}$$

Furthermore, we have

$$\left[qp_b(\rho_{n-1}, \rho_n)^{\alpha} - qp_b(\rho_n, \rho_{n+1})^{1-\alpha}\right]^2 \geq 0 \tag{11}$$

Two cases are recorded from the inequality (11), namely:
(a) $qp_b(\rho_{n-1}, \rho_n)^{\alpha} \leq qp_b(\rho_n, \rho_{n+1})^{1-\alpha}$; and
(b) $qp_b(\rho_{n-1}, \rho_n)^{\alpha} \geq qp_b(\rho_n, \rho_{n+1})^{1-\alpha}$.
From both cases (a) and (b), the following holds:

$$qp_b(\rho_{n-1}, \rho_n)^{\alpha} = qp_b(\rho_n, \rho_{n+1})^{1-\alpha}.$$

Using this equality in (10), we obtain the recurrence relation

$$\begin{aligned} qp_b(\rho_n, \rho_{n+1}) &\le \mu qp_b(\rho_{n-1}, \rho_n)^{2\alpha} \\ &= \mu qp_b(\rho_{n-1}, \rho_n)^{\alpha} qp_b(\rho_n, \rho_{n+1})^{1-\alpha}. \end{aligned}$$

This further gives

$$qp_b(\rho_n, \rho_{n+1}) \le k^n qp_b(\rho_0, \rho_1), \text{ where } k = \mu^{\frac{1}{\alpha}}. \quad (12)$$

The remaining parts follow a routine of the preceding theorem. □

The preceding theorem unifies both the Kannan contraction and the interpolative Kannan-type in quasi-partial b-metric space as follows:

Corollary 2.7. *Let (M, qp_b, s) be a complete quasi-partial b-metric space and $T : M \to M$ be a condensed Kannan-type contraction map in quasi-partial b-metric spaces, for all $\rho, \varrho \in M$ (or $M \backslash Fix(T)$),*

$$qp_b(T\rho, T\varrho) \begin{cases} \le \lambda \left[qp_b(\rho, T\rho) + qp_b(\varrho, T\varrho) \right], \\ \ge \mu qp_b(\rho, T\rho)^{\frac{1}{2}} qp_b(\varrho, T\varrho)^{\frac{1}{2}}, \end{cases} \quad (13)$$

and contrariwise, where $\lambda \in [0, \frac{1}{2s})$ and $\mu \in [0, \frac{1}{s})$ and $s \ge 1$. Then, T has a unique fixed point.

Proof. The proof can be obtained from the preceding theorem if $\alpha = \frac{1}{2}$. □

The following corollary also holds if we drop the interpolative condition in (8).

Corollary 2.8. *Let (M, qp_b, s) be a complete quasi-partial b-metric space and $T : M \to M$ be a condensed Kannan-type contraction satisfying*

$$\begin{aligned} qp_b(T\rho, T\varrho) \le \lambda[&qp_b(\rho, T\rho)^{2\alpha} \\ &+ qp_b(\varrho, T\varrho)^{2(1-\alpha)}], \end{aligned} \quad (14)$$

where $\lambda \in [0, \frac{1}{2s})$, $\alpha \in (0, 1)$ and for all $\rho, \varrho \in M$. Then, T has a unique fixed point in M.

Example 2.9. Let $M = \{a, b, c\}$ and define $qp_b : M \times M \to [0, \infty)$ as follows:

$qp_b(\rho, \varrho)$	a	b	c
a	0	1.0	2.25
b	2.0	1.0	1.25
c	3.75	1.75	1.5

for $\rho, \varrho \in M$. Then, qp_b is a quasi-partial b-metric on M for $s = 1.45$. Indeed, for $(\rho, \varrho) = (a, c)$, then using qp_b4:

$$\begin{aligned} 2.25 &= qp_b(a, c) \\ &< s[qp_b(a, b) + qp_b(b, c)] - qp_b(b, b) \\ &= 1.45(1.0 + 1.25) - 1.0 \approx 2.26. \end{aligned}$$

The function qp_b is neither a metric nor a quasi-partial metric on M.
That is, $qp_b(a, c) \ne qp_b(c, a)$ implies that qp_b is not a metric on M. Also, $qp(a, c) > qp(a, b) + qp(b, c) - qp(b, b)$ shows that qp_b is not a quasi-partial metric on M.
Define the map $T : M \to M$ as:

$$T\rho = \begin{cases} b \text{ for } \rho = a \\ c \text{ for } \rho = b \\ a \text{ for } \rho = c \end{cases}.$$

Then,

i. T is a condensed Kannan-type map on M.

ii. T is not an interpolative Kannan-type map on M.

i. To see this, we set parameters $\alpha = 0.3$, $s = 1.45$, and $(\rho, \varrho) = (a, c)$ in inequality (1). Hence, the terms $qp_b(Ta, Tc)$, $L(\rho, \varrho, \alpha)$, and $U(\rho, \varrho, \alpha)$ are presented, respectively, as follows:

$$\begin{aligned} qp_b(Ta, Tc) &= qp_b(b, a) = 2, \\ \mu L &= \mu qp_b(a, Ta)^{\alpha} qp_b(c, Tc)^{1-\alpha} \\ &= \mu qp_b(a, b)^{0.3} qp_b(c, a)^{0.7} \approx 2.5224\mu, \\ \lambda U &= \lambda\left[qp_b(a, Ta)^{2\alpha} + qp_b(c, Tc)^{2(1-\alpha)}\right] \\ &\le \lambda\left[qp_b(a, b)^{0.6} + qp_b(c, a)^{1.4}\right] \approx 7.3627\lambda. \end{aligned}$$

Observe that these satisfy inequality $2.5224\mu \le 2 \le 7.3627\lambda$, where some choices of $\mu \in (0, \frac{2}{3})$ and $\lambda \in (0, \frac{1}{3})$ fulfill the hypothetical condition. In particular, if $\mu = 0.6$ and $\lambda = 0.3$, then the inequality is $1.513 < 2 < 2.208$.
ii. On the contrary, suppose that T is an interpolative Kannan-type map on M, then $qp_b(Ta, Tc) \le \mu L(\rho, \varrho, \alpha)$. Thus, for the same choice of parameters $\alpha = 0.3$, $s = 1.45$, and $(\rho, \varrho) = (a, c)$, we have $2 < 2.5224\mu$.
This is a contradiction because $\mu > \frac{2}{2.5224} \approx 0.7929 \notin (0, \frac{2}{3})$. Therefore, T is not an interpolative Kannan-type map on M.

3. CONCLUSION

This study introduced a class of condensed Kannan-type contraction mappings within the framework of quasi-partial b-metric spaces in order to investigate the existence properties of nonlinear operators beyond the scope of classical metric structures. By combining the flexibility of quasi-partial b-metric geometry with condensed conditions, both uniqueness and non-unique fixed point results were established for the proposed qp_b-condensed Kannan-type mappings. Furthermore, the study proved the existence theorems of cyclic mappings, namely, trivial and nontrivial cyclic condensed Kannan-type contractions. The obtained results generalize several existing fixed point theorems and provide a unified framework that broadens the applicability of contraction principles in more generalized settings. To support the theoretical results, an illustrative example was presented to demonstrate the advantage of the proposed concepts over other related works in the literature. Overall, the study contributes to the ongoing development of fixed point theory by extending the notions of Kannan contraction, interpolative Kannan-type contraction, condensed Kannan-type contraction, and asymmetry distance frameworks.

DATA AVAILABILITY

No datasets were generated or analyzed during the current study.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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