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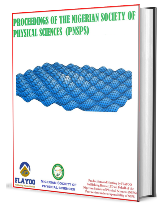

Influence of a topological defect on energy and partition function for a Yukawa potential

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ABSTRACT

This study obtains the solution of the Schrödinger equation for a Yukawa potential interacting with a cosmic string as a topological defect (TD). The calculated energy levels are used to obtain the partition function (Z) under the TD. Some thermodynamic properties such as enthalpy (H), entropy (S) and Gibbs free energy (G), are studied under the influence of the TD. The results are presented for systems with and without the defect. They show that the presence of the TD consistently increases the energy eigenvalue as a function of both the potential strength and the screening parameter. The presence of the TD lowers S, increases Z and G, and has no effect on H as a function of temperature. The defect also increases Z and G as functions of the potential strength. Generally, the TD affects the energy, Z, S and G but has no effect on H because the topological parameter cancels out during computation.

Keywords: Eigensolutions, Bound state, Thermodynamic properties, Potential model.

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1. INTRODUCTION

The development of quantum mechanics due to discrete energy spectra is tied to the generation of potential models for theoretical framework [1–4]. These potential models provide detail understanding of bound states in areas where classical physics fails to provide comprehensive information. This development leads to the emanation of wave equation, culminating in Schrödinger equation especially where interactions are represented via potential models. The interactions within quantum particles are important means of validating quantum theory by accurately reproducing atomic spectra [5]. A continuous development leads to more formulation of potential models that represent the realistic interactions either short-range or long-range [6].

For instance, Yukawa potential as a short-range potential, was designed by Hideki Yukawa in 1935 to describe nuclear forces carried by massive particles which links quantum theory with nuclear physics to enhance the theoretical landscape [7, 8]. The potential models also offered practical tools for testing approximation and numerical approaches which serve as the foundation of modern quantum traditional techniques as the offer clear descriptions of complex many-body systems. They shaped studies of important areas such as quantum thermodynamics, information theory and topological phenomena [9].

A topological phenomenon exists as TD in the form of cosmic string [10, 11]. It featured as a purely topological perturbation which affects the effective angular momentum through the conical parameter, leading to energy-level shifts, lifting degeneracy and modifications of thermodynamic quantities [12, 13]. A cos-

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mic string introduces a conical spacetime geometry as metric remains flat, changes the global structure, encode the angular momentum number via a topological parameter, characterized by a deficit angle, making the Schrödinger equation remains solvable with a clear modification of the angular momentum term. These characteristics have made this TD interesting and has been applied for other studies [14–16].

Other defects such as monopoles requires the introduction of nontrivial gauge fields that makes it difficult and impossible to obtain exact analytical solutions of the Schrödinger equation. Many defects excluding cosmic string, do not provide a straightforward mechanism for studying degeneracy breaking. Motivated by the interest on the effect of TD as cosmic string on thermodynamic properties and the importance of Yukawa potential model, this study wants to examined the effect of TD on the energy level and thermodynamic properties of Yukawa potential. The shot-range Yukawa potential model is given by:

$$V(r) = -\frac{Ae^{-ar}}{r}, \tag{1}$$

where A is the potential strength standing as the depth of the potential and α is the screening parameter that sets the effective range. The potential decays exponentially, making it suitable for modeling finite-range forces mediated by massive particles. The Yukawa potential has been applied to study different systems [17]. It has been studied for bound states in the non-relativistic system [18, 19]. Under the Dirac equation, it was employed as tensor interaction [20]. It was also adopted in the interaction of nuclear forces. On a broad term, Yukawa potential remains a versatile and widely used model for exploring short-range interactions, bound-state dynamics, and thermodynamic properties across physics. Thermodynamic properties analysis the macroscopic character and link the microscopic particle with macroscopic properties via statistical mechanics [21–26]. The thermodynamic properties such as H, S, G and heat capacity, are derived from Z which serves as the bed rock of the thermodynamic properties.

2. SOLUTIONS OF THE SCHRÖDINGER EQUATION

A radial Schrödinger equation with energy level E, potential model $V(r)$, wave functions $R(r)$, reduced Planck's constant $\hbar$ and reduced mass μ, is given as:

$$-\frac{\hbar^2}{2\mu}\frac{d^2R(r)}{dr^2} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}R(r) = ER(r) - V(r)R(r). \tag{2}$$

The angular momentum number will be modified by a single constant m^2 due to the presence of the TD and induced a topological parameter η^{-2}. The cosmic string introduces a conical spacetime where the space is locally flat but globally has deficit angle. The deficit angle is encoded through a topological parameter η such that the angular coordinate is effectively rescaled; $\emptyset \in [0, 2\pi\eta]$ rather than $[0, 2\pi]$. Because the angular coordinate is rescaled, the angular derivative changes. However, the present study considered the effective angular momentum as $\ell(\ell+1) \to \ell_{\text{eff}}(\ell_{\text{eff}}+1)$. With the angular momentum number been modified by a single constant due to the presence of TD, then, $\ell(\ell+1) = \frac{m^2}{\eta^2}$. The centrifugal term will be approximated by the formula [27, 28] $\frac{\alpha^2}{(1-e^{-ar})^2}$. This approximate is valid for $\alpha \ll 1$. Define $y = e^{-ar}$, and plugging equation (1) into equation (2) leads to:

$$R''(y) + y^{-1}R'(y) + \frac{\lambda_0(1-y)^2 + \lambda_1 y(1-y) - m^2\eta^{-2}}{y^2(1-y)^2}R(r) = 0. \tag{3}$$

Recall the reference equation for the methodology of Nikiforov-Uvarov method [29]:

$$\psi''(y) + \frac{\overline{r}(y)}{\sigma(y)}\psi'(y) + \frac{\overline{\sigma}(y)}{\sigma^2(y)}\psi(y) = 0. \tag{4}$$

Comparing equation (3) and equation (4) gives the following relations $\sigma^2(y) = y^2(1-y)^2, \overline{\sigma}(y) = \lambda_0(1-y)^2 + \lambda_1 y(1-y) - m^2\eta^{-2}$. After much algebraic simplifications, we have:

$$\frac{\mu A}{\alpha(n+\lambda_2)} = \frac{n+\lambda_2}{2} + \sqrt{m^2\eta^{-2} - \frac{2\mu E}{\alpha^2}}, \tag{5}$$

where $\lambda_2 = \frac{1}{2}\left(1 + \sqrt{4m^2\eta^{-2}+1}\right)$. Further evaluation of equation (5) gives the energy levels as:

$$E_{n,m} = \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{m^2}{\eta^2} - \left[\frac{2\mu A}{\alpha\left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1+\left(\frac{2m}{\eta}\right)^2}\right)} - \frac{n + \frac{1}{2} + \frac{1}{2}\sqrt{1+\left(\frac{2m}{\eta}\right)^2}}{2}\right]^2\right]. \tag{6}$$

2.1. PARTITION FUNCTION (Z)

At this point, we calculate the canonical Z with cosmic string as TD. In the canonical Z here, there are double summation with one over the quantum number and the other over the modification on the rotational quantum number. Therefore,

$$Z(\beta) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} e^{-\beta E_{n,m}}, \quad \beta = \frac{1}{k_\beta T}, \tag{7}$$

where k_β is the Boltzmann constant and T is temperature. Considering the two parts, the energy levels of equation (6) can be split as:

$$E_{n,m}^{v} = \frac{\alpha^2\hbar^2}{2\mu}\left(\frac{2\mu A}{\alpha\lambda_3} - \frac{\lambda_3}{2}\right)^2, \quad E_m = \frac{\alpha^2\hbar^2}{2\mu}\frac{m^2}{\eta^2},$$
$$\lambda_3 = n + \frac{1}{2} + \frac{1}{2}\sqrt{1+\left(\frac{2m}{\eta}\right)^2}. \tag{8}$$

Therefore, equation (7) can be written as:

$$Z(\beta) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} e^{-\beta E_m} e^{\beta E_{n,m}^{v}}. \tag{9}$$

To solve equation (9) fully, we replace the summation with integral. For larger m or quasi-continuous,

$$Z(\beta) = \int_{-\infty}^{\infty} e^{-\beta E_m} dm \int_0^{\infty} e^{\beta E_{n,m}^{v}} dn. \tag{10}$$

Plugging in the values of the energies into equation (10) and simplify the integrals gives a complete canonical Z as:

$$Z(\beta) = \frac{\eta}{\alpha\hbar}\sqrt{\frac{2\mu\pi}{\beta}}\exp\left[\frac{\beta\alpha^2\hbar^2}{2\mu}\left(\frac{4\mu A}{3\alpha\hbar^2} - \frac{3}{4}\right)^2\right]. \tag{11}$$

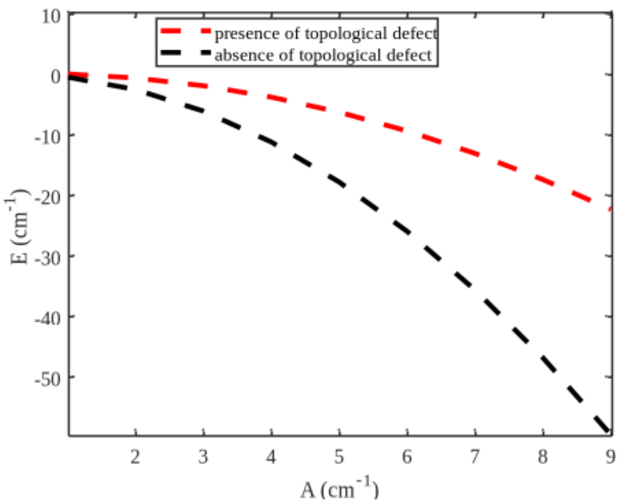


Figure 1. Effects of the potential strength on the energy levels in the presence and absence of the TD at the ground state, with $\mu = \hbar = m = 1$ and $\alpha = 0.25$.

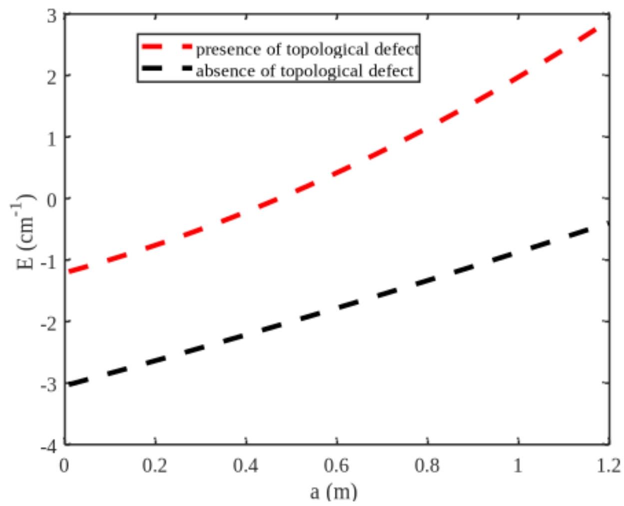


Figure 2. Effects of the screening parameter on the energy levels in the presence and absence of the TD at the ground state, with $\mu = \hbar = m = 1$ and $A = 2$.

2.2. ENTHALPY

Enthalpy is given as $H = U + PV$, for ideal system, $PV = Nk_\beta T$, but in this context (single molecule or quantum system), $V = 0$. Thus,

$$H = U = -\frac{\partial}{\partial \beta} \ln Z(\beta). \quad (12)$$

Plugging equation (11) into equation (12) gives a complete enthalpy as:

$$H = \frac{k_\beta T}{2} - \frac{\alpha^2 \hbar^2}{2\mu}\left(\frac{4\mu A}{3\alpha\hbar^2} - \frac{3}{4}\right)^2. \quad (13)$$

2.3. ENTROPY

Entropy S is given as:

$$S = k_\beta \left[\ln Z(\beta) + \beta U\right]. \quad (14)$$

Plugging equation (11) and equation (13) into equation (14) gives:

$$S = \frac{k_\beta}{2}\left[1 + \ln\left(\frac{2\mu\pi\eta^2 k_\beta T}{\alpha^2\hbar^2}\right)\right]. \quad (15)$$

2.4. GIBBS FREE ENERGY

Gibbs free energy G is given as:

$$G = -k_\beta T \ln Z(\beta). \quad (16)$$

Plugging equation (11) into equation (16) gives the full G as:

$$G = \frac{\alpha^2\hbar^2}{2\mu}\left(\frac{4\mu A}{3\alpha\hbar^2} - \frac{3}{4}\right)^2 - \frac{k_\beta T}{2}\ln\left(\frac{2\mu\pi\eta^2 k_\beta T}{\alpha^2\hbar^2}\right). \quad (17)$$

3. RESULTS AND DISCUSSION

Figure 1 shows how the bound-state energy at the ground state changes with the coupling strength A, both with and without the topological defect. As A increases, the energy levels shift monotonically, indicating stronger binding due to the attractive part of the potential. The curve corresponding to the TD (cosmic string) is systematically displaced relative to the defect-free case. Physically, this comes from the conical geometry induced by the defect, which effectively rescales the angular momentum contribution and modifies the centrifugal barrier. As a result, the defect alters the localisation of the wave function and hence the energy spectrum, even though the potential itself is unchanged. In short, increasing A deepens the bound states, while the TD introduces an additional geometric shift in the energies.

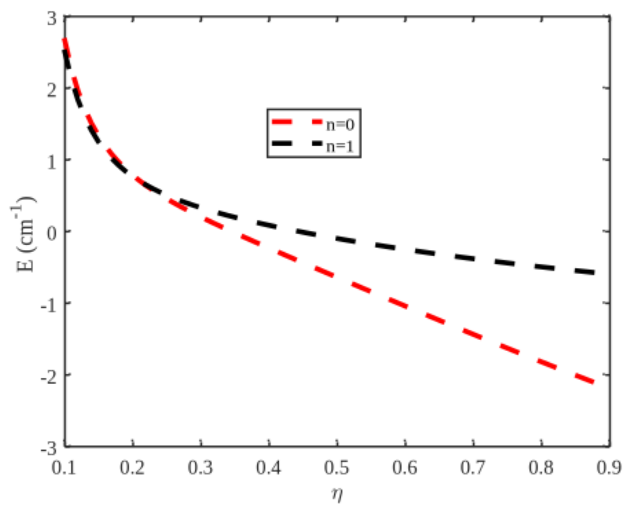
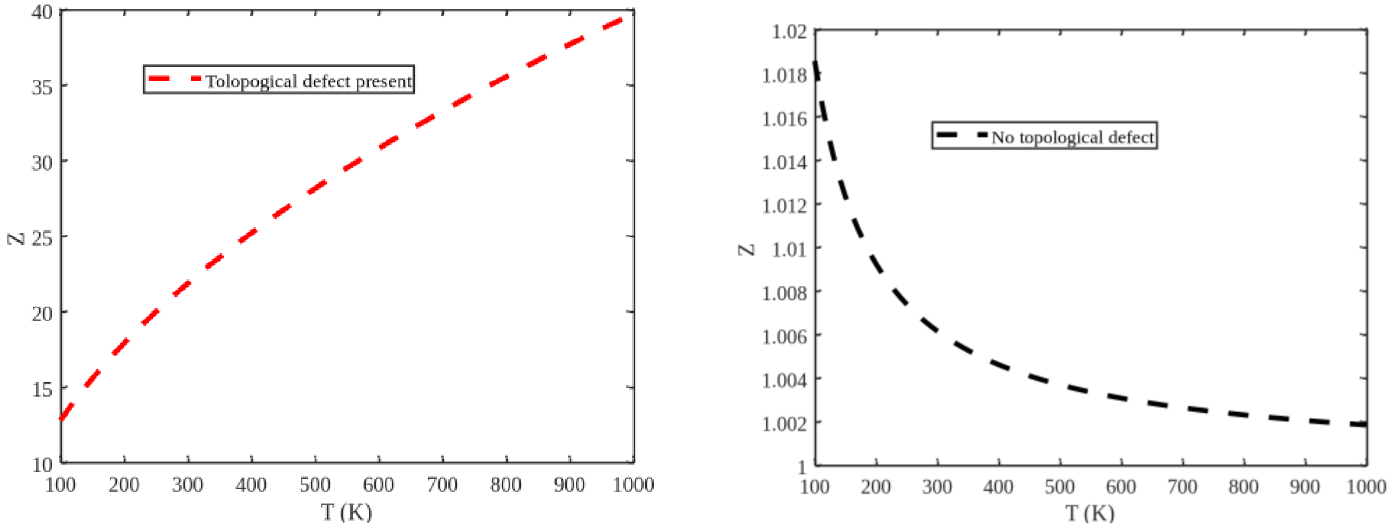


Figure 3. Variation of the topological parameter with the energy levels at the ground state and first excited state, with $\mu = \hbar = m = 1$, $\alpha = 0.25$ and $A = 2$.

Figure 2 shows the effects of the screening parameter on the energy levels in the presence and absence of the TD at the ground state. Here, the screening parameter controls how short-ranged the interaction is. Increasing α weakens the effective attraction (stronger screening), so the energy levels move upward towards less tightly bound states. Again, the presence of the TD consistently shifts the spectrum compared with the ordinary-space case. This shows that screening and topology act in different ways: α changes the range of the force, while the defect modifies the ge-



(a) In the presence of the TD.

(b) In the absence of the TD.

Figure 4. Variation of the partition function against temperature.

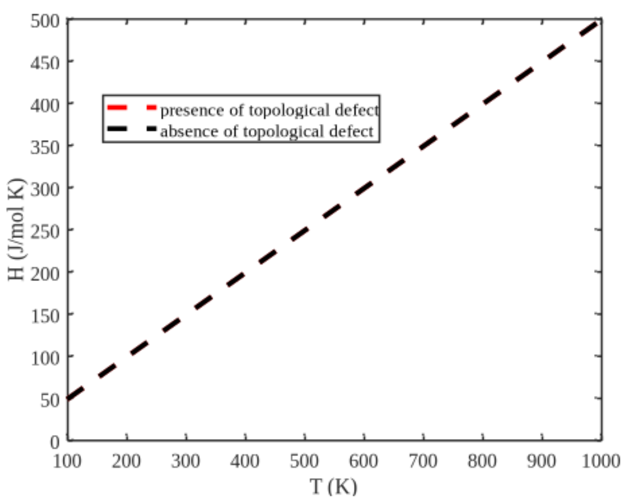


Figure 5. Variation of H against temperature.

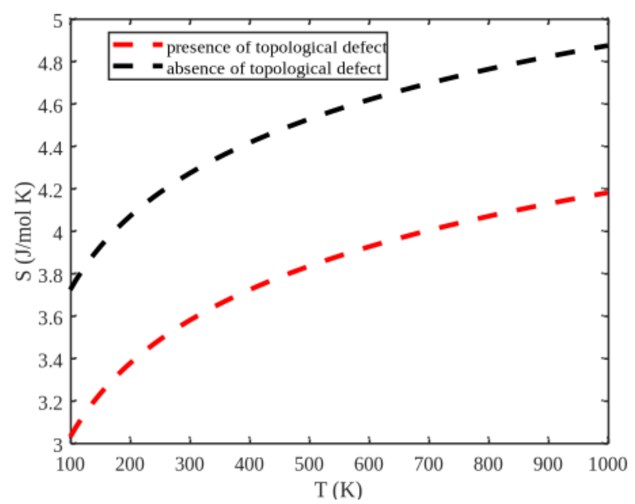


Figure 6. Variation of entropy against temperature with and without the TD.

ometry felt by the particle. Thus, even for the same screening strength, the defected system supports different bound energies.

Figure 3 depicts the variation of the topological parameter with the energy levels at the ground state and first excited state. As η varies, both energy levels change smoothly. The excited state always lies above the ground state, but both respond similarly to the geometric deformation. This confirms that the cosmic string acts as a controllable external parameter that tunes the entire spectrum. The key point is that topology alone, even without changing A or α, can significantly modify quantum energies.

Figure 4(a) presents Z against temperature in the presence of the TD, while Figure 4(b) presents Z against temperature without the TD. At low temperatures, only the ground state contributes significantly, leading to small Z with the TD. As temperature rises, the Boltzmann factor becomes less suppressive because the excited states start contributing, making more terms participate effectively in the sum. This, in turn, increases the partition function. However, in the absence of the TD, the sum replaced by the finite number of levels is dominated by the negative bound state. Because $E_n < 0$, an increase in temperature reduces the statistical weight of the state, leading to a reduction in Z.

Figure 5 depicts the relationship between H and temperature. The enthalpy increases linearly with increasing temperature. It is unaffected by the TD. This follows from the analytic equation for H in equation (13), where the cosmic-string parameter is completely annulled during simplification.

Figure 6 shows the effect of temperature on S with and without the TD. The entropy increases as the temperature increases for both the presence and absence of the TD. A rise in temperature makes higher energy states thermally accessible and increases the number of available microstates. This enhances disorder in the system, leading to monotonic growth of S. However, S obtained

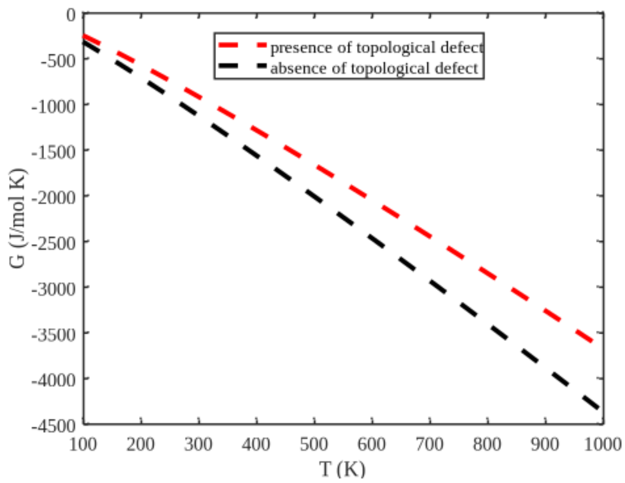


Figure 7. Variation of *G* against temperature with and without the TD.

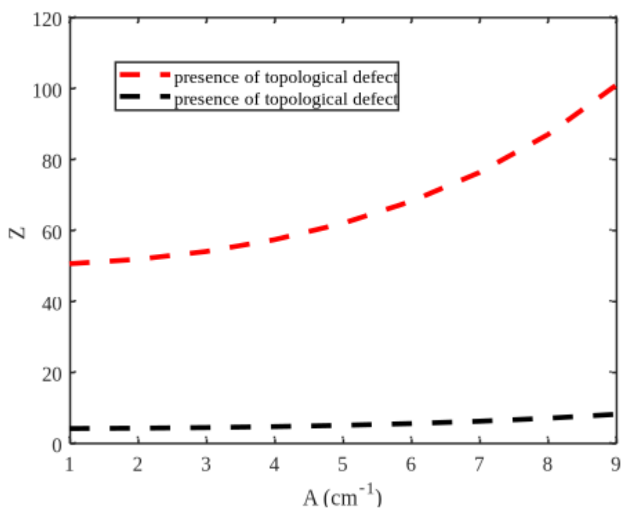


Figure 8. Variation of the partition function against the potential strength *A* with and without the TD.

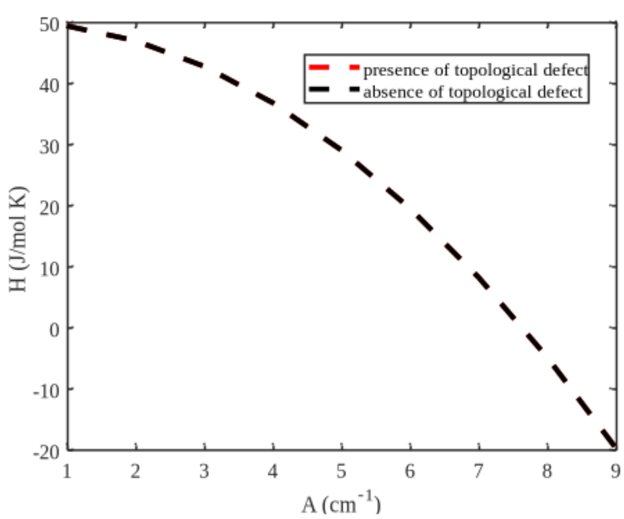


Figure 9. Enthalpy versus the potential strength with and without the TD.

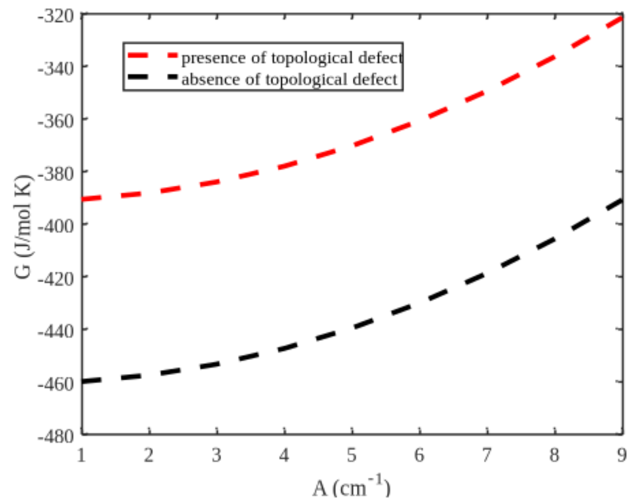


Figure 10. Variation of Gibbs free energy against the potential strength *A* with and without the TD.

in the presence of the TD is lower than S obtained in its absence at all temperatures. The consistently lower S in the presence of the TD arises from the geometric deformation introduced by the defect. The defect creates a conical spacetime characterised by a topological parameter that reduces angular symmetry and curtails the degeneracy of quantum states. This allows fewer available microstates and less disorder compared with the absence of the TD. Generally, the TD acts as a constraint on particle motion, tightening the level spacing and suppressing state multiplicity.

Figure 7 displays G against temperature in the presence and absence of the TD. The G decreases with increasing temperature in both cases. The decrease in G with increasing temperature follows directly from thermodynamics, where $G = H - ST$. This means that as temperature increases, the term ST increases and thus reduces G. However, G obtained in the presence of the TD is higher than G obtained without the TD at all the temperatures considered. The consistently higher G in the presence of the TD is due to the modification of the energy spectrum by the conical geometry. The presence of the defect reduces state degeneracy and restricts the available phase space. This alters the internal energy and lowers entropy compared with the absence of the defect. Since the presence of the defect diminishes S, the subtractive term ST is reduced. This shows why G with topology is higher than G without topology. Generally, the TD acts as an external constraint on the system, making it thermodynamically less favourable and effectively raising its free-energy cost.

Figure 8 depicts the effect of the potential strength A on Z with and without the TD. The Z increases as the potential strength A increases. The Z with the TD remains higher at every temperature compared with Z without the TD. The presence of the defect greatly influences Z, as shown by the wide margin between Z with the defect and Z without the defect. This is because the presence of the TD modifies the energy and the density of states on which Z depends exponentially. A small change therefore results in a large increase.

Figure 9 shows the relationship between H and the potential strength in the presence and absence of the TD. An increase in the potential strength results in a monotonic decrease in H. However, the TD does not have any effect on H. This is evident from equation (13), where the topological parameter does not appear. After the computation of H, the TD cancels out, reflecting its non-effect on this property for the present potential model.

Figure 10 presents the effect of the potential strength on G with and without the TD. The G increases as the potential strength A increases. The G with the TD remains higher at every temperature compared with G without the TD. As the potential strength increases, the potential becomes stronger, reflecting less energetic availability of the system, which in turn increases G.

From the foregoing, the TD in the cosmic string furnishes a mathematically tractable but physically robust framework that enables tunable geometric modifications, lifts degeneracy, systematically shifts energy spectra and provides clear thermodynamic signatures without losing analytical accessibility, even while acting as a constraint on some thermodynamic properties.

4. CONCLUSION

In this study, the presence of the TD as a cosmic string on the energy level of a short-range Yukawa potential is analysed. The influence of the partition function and other thermodynamic properties is critically examined using the analytic equations and graphical results. The presence of the cosmic string shifts energy levels, lifts degeneracies and reduces entropy while leaving enthalpy unaffected. The findings reflect the behaviour of the TD as a geometric constraint on quantum systems, changing both microscopic bound-state characteristics and macroscopic thermodynamic characteristics. This study also highlights the crucial connections among potential strength, the screening parameter and topological geometry, demonstrating the role of topology in controlling energy spectra and thermodynamic variables.

DATA AVAILABILITY

The data will be available on request from the corresponding author.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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