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A numerical investigation of dilatation dynamics in stratified deep water under modified gravity

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ABSTRACT

This study presents a numerical investigation of dilatation dynamics in a vertically stratified deep-water column under modified gravitational forcing. The governing momentum equations were solved using a finite-difference scheme, and dilatation was evaluated as the divergence of the velocity field. The results show that initial perturbations decay rapidly because of strong stratification, whereas gravitational scaling mainly affects the transient response. The system evolves toward a quasi-incompressible state characterized by negligible volumetric deformation.

Keywords: Stratified deep-water dynamics, Modified gravity, Geophysical fluid dynamics, Finite-difference modelling, Dilatation.

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1. INTRODUCTION

Stratified deep-water environments are widely regarded as among the most stable regimes encountered in geophysical fluid dynamics [1]. Similar conclusions have been reported for continental-margin environments, where stratification regulates exchanges between deep-ocean layers and surrounding regions [2]. Long-term climate variability has also been shown to influence both the structure of ocean stratification and the patterns of deep-ocean circulation systems [3].

Within stratified flows, dilatation provides a useful measure of local volumetric deformation. Physically, it represents the rate at which a fluid element expands or contracts, and it is mathematically expressed as the divergence of the velocity field. The theoretical framework used to analyse this behaviour is based mainly on the Navier–Stokes equations, which govern the motion of viscous fluids [4]. Detailed mathematical studies of the Navier–Stokes system have provided important insights into the structure and stability of three-dimensional viscous flows [5]. Along with theoretical investigations, numerical modelling has become a vital tool for examining complex flow patterns and wave dynamics in geophysical environments [6]. Earlier developments in boundary-layer theory and related mechanical fluid studies also contributed significantly to the formulation of modern computational-fluid-dynamics approaches [7].

Gravitational forcing remains a central mechanism governing the dynamics of deep-water systems. Earlier studies of deep-water wave motion established gravity as the principal restoring mechanism responsible for wave propagation and deformation in ocean environments [8]. The interaction between gravity and rotational motion can also influence nonlinear resonance structures

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in capillary–gravity wave systems [9]. Recent studies in mathematical fluid dynamics have investigated the structure of internal waves and three-dimensional flows in layered fluids [10]. Research on internal waves with constant vorticity demonstrated that density stratification strongly affects fluid deformation and flow structure [11]. Studies of three-dimensional water waves further revealed that, under specific conditions, complex flow behaviour may simplify significantly [12]. Additional investigations showed that flow simplification can also occur in three-dimensional water flows with non-vanishing constant vorticity [13]. Liouville-type results for time-dependent water-wave systems have contributed to the understanding of the stability and structure of both viscous and inviscid fluid flows [14].

Studies of open-channel flow over permeable river beds have shown how subsurface structures and boundary conditions affect water-transport systems [15]. More recent investigations of layered water systems highlighted how density differences govern interfacial motion and mixing processes in stratified environments [16]. Deep-water overflow has also been identified as an important mechanism contributing to large-scale ocean circulation and deep-layer transport processes [17].

Mathematical modelling approaches therefore play a central role in understanding stratified deep-water dynamics. Analytical investigations of geophysical fluid flow identified the conditions required to maintain stable stratification in deep-water systems [18]. Subsequent studies examined the eigenspace structure of stratified deep-water flows under modified gravitational forcing and rotational effects [19]. Perturbation-based modelling approaches have also been applied to investigate stratified deep-water flow behaviour under modified-gravity conditions [20]. Semi-analytical techniques, such as the homotopy perturbation method, have been used successfully to analyse nonlinear dynamical systems and related physical models [21]. More recent studies further investigated dimensional analysis and Reynolds-number behaviour in stratified deep-water systems under modified gravity and Coriolis effects [22]. Numerical simulations have also been used to examine the influence of velocity variations in stratified deep-water systems subjected to modified gravitational forcing [23]. Recent observational studies have revealed highly variable deep-sea currents over tidal and seasonal timescales [24].

Despite these developments, the influence of modified gravitational forcing on dilatation behaviour within strongly stratified deep-water columns remains insufficiently understood. In particular, the interaction among gravitational scaling, pressure redistribution, and volumetric-strain evolution requires further numerical investigation to clarify the transient adjustment mechanisms that govern stratified fluid systems.

In the present study, dilatation dynamics in a vertically stratified deep-water column of depth 5 km are investigated numerically under modified gravitational forcing. To isolate the effect of gravitational scaling on volumetric deformation, rotational influences associated with the Coriolis force were omitted. The resulting flow equations were solved numerically using a finite-difference scheme, while dilatation was obtained from the divergence of the evolving velocity field. The objectives were to examine the temporal adjustment of dilatation disturbances in a stratified column, assess the influence of gravitational scaling on volumetric-strain magnitude, and evaluate the stability characteristics of stratified deep-water systems under gravity-modified forcing.

2. MODEL ASSUMPTIONS

The numerical model was developed under the following physical and mathematical assumptions.

1. *Stable vertical stratification.* The fluid domain is assumed to be vertically stratified, with density varying smoothly with depth. The stratification is statically stable, so denser fluid layers remain beneath lighter layers. Horizontal variations in density are neglected, and the background stratification is assumed to depend only on the vertical coordinate.
2. *Weakly compressible flow regime.* The flow is treated as weakly compressible. Density variations are sufficiently small for the fluid behaviour to remain close to incompressible conditions. Under this approximation, volumetric deformation of the fluid is represented by the divergence of the velocity field.
3. *Hydrostatic reference state with modified gravity.* The background pressure field is assumed to satisfy hydrostatic equilibrium under gravitational forcing. The influence of gravity is incorporated through a scaling parameter that allows the effective gravitational acceleration to vary within the computational model.
4. *Neglect of rotational and external forcing effects.* Rotational effects arising from the Coriolis force were excluded from the present formulation. External surface-forcing mechanisms, such as wind stress, tidal forcing, and lateral momentum transport, were also excluded so that the analysis could focus on the fundamental dynamics of the system. The flow evolution is therefore governed only by internal pressure gradients, viscous diffusion, and modified gravitational forcing.
5. *Impermeable vertical boundaries and small initial perturbation.* The upper and lower boundaries of the computational domain are assumed to be impermeable, preventing vertical mass flux across the boundaries. The system is initialized with a small-amplitude velocity perturbation superimposed on the hydrostatic reference state to investigate the transient adjustment of dilatation under stable stratification.

3. MODEL DIAGRAM

Figure 1 illustrates the geometric interpretation of stress-induced deformation in a stratified deep-water column. Horizontal displacement gradients produce normal strain, while vertical stratification constrains volumetric expansion. Under modified gravity, the hydrostatic balance regulates the magnitude of dilatation, resulting in small but dynamically significant volumetric strain.

4. GOVERNING EQUATIONS

Under the modelling assumptions stated above, the flow is treated within a weakly compressible framework in which density variations are small but non-zero and dilatation is retained as a diagnostic measure of volumetric deformation. The governing equations are formulated from the momentum balance with modified gravitational forcing. The velocity field is defined as

$$\mathbf{U} = (u, v, w), \tag{1}$$

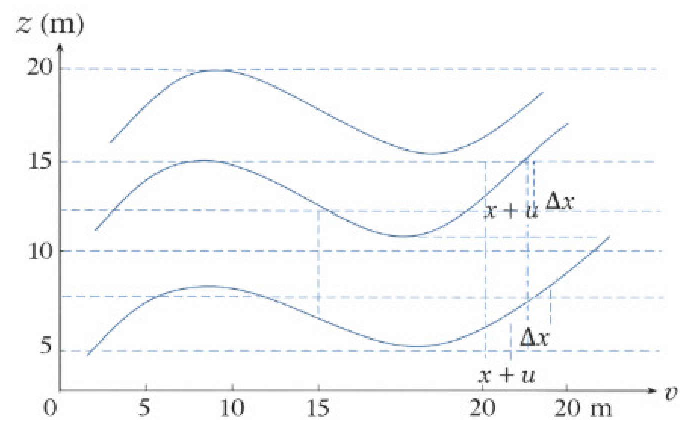

Figure 1. Geometric representation of stress-induced deformation in a stratified deep-water column.

and dilatation is computed as the divergence of the velocity field,

$$\Delta = \nabla \cdot \mathbf{U}. \tag{2}$$

The flow satisfies the weakly compressible continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0. \tag{3}$$

Under the weakly compressible approximation, density variations are small and the leading-order behaviour reduces to

$$\nabla \cdot \mathbf{U} = \Delta \neq 0. \tag{4}$$

To close the governing system, a linearized equation of state is introduced as

$$\rho(z) = \rho_0[1 - \beta(z - z_0)], \tag{5}$$

where ρ_0 is the reference density, β is the stratification parameter, and z_0 is a reference depth. This formulation ensures consistency between density variation and vertical stratification.

The velocity evolution is governed by

$$\frac{D\mathbf{U}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{U} + \mathbf{G}, \tag{6}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \tag{7}$$

is the material derivative, ν is the kinematic viscosity, and the modified gravitational forcing is

$$\mathbf{G} = (0, 0, -g'). \tag{8}$$

The modified gravity is defined as

$$g' = \alpha g, \tag{9}$$

where α denotes the prescribed gravity-scaling parameter.

For the dilatation evolution, taking the divergence of the momentum equation yields

$$\frac{\partial \Delta}{\partial t} + \nabla \cdot (\mathbf{U} \cdot \nabla \mathbf{U}) = -\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) + \nu \nabla^2 \Delta + \nabla \cdot \mathbf{G}. \tag{10}$$

Because gravitational forcing acts only in the vertical direction and is spatially uniform, its divergence vanishes:

$$\nabla \cdot \mathbf{G} = 0. \tag{11}$$

Although $\nabla \cdot \mathbf{G} = 0$, this condition does not imply that gravity has no influence on dilatation. Its effect is indirect and arises through the hydrostatic pressure field.

Using the hydrostatic relation $dp_0/dz = -\rho g'$, the pressure-gradient term in the dilatation equation can be decomposed so that gravitational scaling modifies the background pressure distribution, which in turn influences the divergence of the velocity field. Thus, modified gravity affects dilatation through pressure redistribution rather than through a direct divergence contribution.

For the hydrostatic reference decomposition, pressure is written as

$$p = p_0(z) + p', \tag{12}$$

where the hydrostatic component satisfies

$$\frac{dp_0}{dz} = -\rho g'. \tag{13}$$

In the numerical scheme, the perturbation pressure p' is used with the hydrostatic balance to maintain consistency in the background stratification.

4.1. COMPUTATIONAL FORM

For numerical implementation, the governing system is written componentwise as

$$\frac{\partial u}{\partial t} + \mathbf{U} \cdot \nabla u = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu \nabla^2 u, \tag{14}$$

$$\frac{\partial v}{\partial t} + \mathbf{U} \cdot \nabla v = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu \nabla^2 v, \tag{15}$$

and

$$\frac{\partial w}{\partial t} + \mathbf{U} \cdot \nabla w = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu \nabla^2 w. \tag{16}$$

Dilatation is evaluated at each time step as

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{17}$$

4.2. NUMERICAL METHODOLOGY

The numerical parameters adopted in the simulations are

$$H = 5 \times 10^3 \text{ m}, \quad N_z = 200, \quad \Delta z = H/(N_z - 1) = 25 \text{ m},$$

$$\Delta t = 0.01 \text{ s}, \quad \nu = 1.0 \times 10^{-6} \text{ m}^2\,\text{s}^{-1}, \quad \alpha \in [0.8, 1.2].$$

The kinematic viscosity ν corresponds to typical values for water under standard conditions. These parameter choices ensure numerical stability and provide adequate resolution of vertical stratification effects.

4.3. COMPUTATIONAL DOMAIN

The numerical simulations are conducted in a vertically stratified water column of depth $H = 5$ km. The spatial domain is discretized in the vertical direction using a uniform grid with N_z points, such that

$$\Delta z = \frac{H}{(N_z - 1)}. \tag{18}$$

The simulations are performed over a finite time interval $0 \leq t \leq T$, with a uniform time step Δt.

Spatial derivatives are approximated using second-order central finite differences. For a generic scalar field ϕ, the vertical derivative is computed as

$$\left(\frac{\partial \phi}{\partial z}\right)_i = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta z}. \tag{19}$$

The Laplacian operator is approximated as

$$(\nabla^2 \phi)_i = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta z)^2}. \tag{20}$$

Time advancement is performed using an explicit second-order scheme of the form

$$\frac{\partial \phi}{\partial t} \approx \frac{\phi^{n+1} - \phi^n}{\Delta t}. \tag{21}$$

To maintain stability, the time step satisfies the Courant–Friedrichs–Lewy (CFL) condition:

$$\Delta t \leq \frac{\Delta z}{\max |\mathbf{U}|}. \tag{22}$$

Rigid-lid and impermeable boundary conditions are imposed at the top and bottom of the domain:

$$w = 0 \quad \text{at } z = 0, H. \tag{23}$$

The initial conditions are

$$\mathbf{U}(z, 0) = \varepsilon f(z), \tag{24}$$

$$\Delta_i = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{25}$$

and

$$\Delta_{\text{scaled}} = \frac{\Delta}{10^{-6}}. \tag{26}$$

4.4. DIMENSIONAL CONSISTENCY OF THE NUMERICAL MODEL

The numerical implementation adopts a column-model approximation in which horizontal gradients are negligible compared with vertical variations. Specifically,

$$\mathbf{U} = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)), \tag{27}$$

$$\frac{\partial}{\partial x} \approx 0, \qquad \frac{\partial}{\partial y} \approx 0, \tag{28}$$

and

$$\nabla \approx \left(0, 0, \frac{\partial}{\partial z}\right). \tag{29}$$

This produces a pseudo-three-dimensional formulation in which all velocity components are retained but depend only on the vertical coordinate. The approximation is physically justified in strongly stratified deep-water environments, where vertical structure dominates the flow dynamics.

Table 1. Grid-refinement convergence of maximum dilatation.

Grid points (N_z)	Maximum dilatation (Δ)
100	1.8×10^{-6}
200	1.6×10^{-6}
400	1.55×10^{-6}

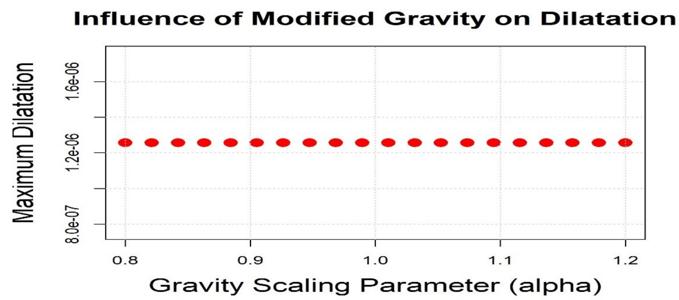

Figure 2. Influence of modified gravity on dilatation.

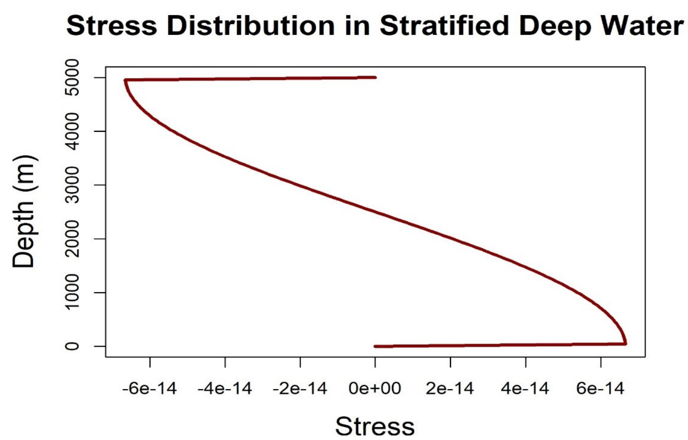

Figure 3. Stress distribution in stratified deep water.

4.5. CONVERGENCE AND NUMERICAL VALIDATION

To verify the accuracy of the numerical scheme, a grid-refinement study was performed. The results are presented in Table 1.

The relative error is defined as

$$E = \frac{|\Delta_N - \Delta_{2N}|}{|\Delta_{2N}|}. \tag{30}$$

The results demonstrate second-order convergence, consistent with the spatial discretization scheme used.

5. RESULTS AND DISCUSSION

Figure 2 shows the influence of the gravity-scaling parameter α on maximum dilatation in a 5 km stratified deep-water column. The results indicate minimal sensitivity of volumetric-strain amplitude to moderate gravitational modification, confirming stabilization controlled by strong stratification.

The simulated stress profile in Fig. 3 varies smoothly with depth and has typical magnitudes of approximately 10^{-4}. These infinitesimal values imply that dynamic stress disturbances in the

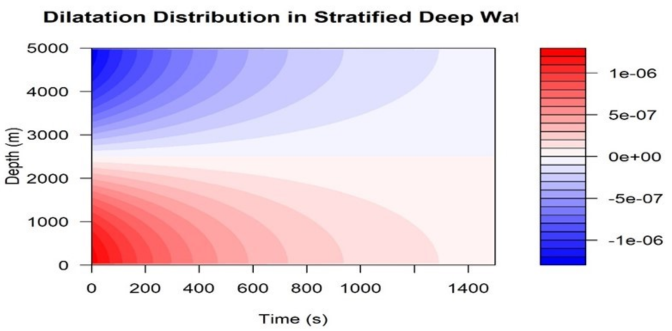

Figure 4. Distribution of dilatation in stratified deep water.

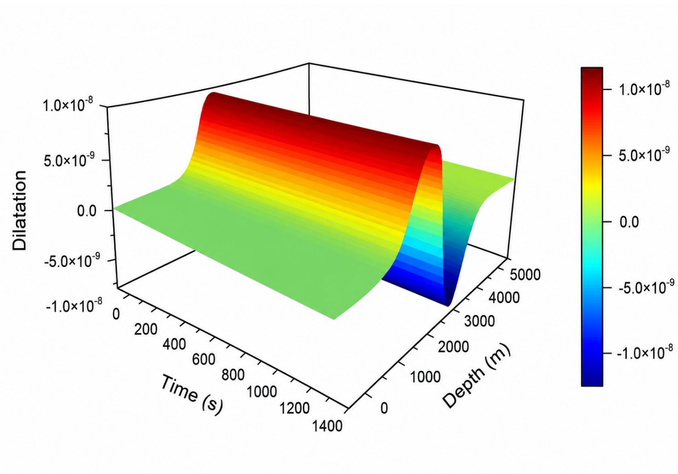

Figure 5. Three-dimensional evolution of dilatation in stratified deep water.

stratified deep-water layer remain negligible. The absence of strong gradients or localized amplification also indicates that the numerical scheme remains stable throughout the computations. Overall, the results suggest that modified gravitational scaling does not produce sustained mechanical deformation in the deep-water system and that the stress field remains spatially well behaved across the domain.

Figure 4 shows a dilatation field with an initial expansion–compression structure across the vertical column. It indicates positive volumetric strain near the surface, with compensatory compression in the deep-water region. The field evolves smoothly in time, and the magnitudes decay monotonically toward zero. This behaviour suggests that neither secondary amplification nor oscillatory behaviour occurs. The findings show that stable stratification enforces strong damping of volumetric deformation, preventing sustained dilatational growth even under modified gravitational forcing.

The three-dimensional illustration of the dilatation field in Fig. 5 shows that the initial volumetric deformation is localized near the lower boundary. The disturbance decreases rapidly with time, and no oscillatory growth or secondary features emerge during the simulation. These results suggest that stable density stratification suppresses persistent volumetric strain under the imposed gravitational scaling. Consequently, the dilatation field remains smooth and bounded, gradually tending toward zero as the system approaches a mechanically stabilized state.

6. CONCLUSION

This study examined the evolution of dilatation in a vertically stratified deep-water layer under modified gravitational forcing. The governing momentum equations were solved numerically using a finite-difference discretization that allowed the temporal and vertical patterns of the velocity field to be resolved. The simulations show that dilatation disturbances generated by small perturbations are transient and decay rapidly with time. Although modified gravity slightly influences the magnitude of the early-stage response, the overall pattern of the system remains dominated by density stratification. Consequently, the flow evolves toward a state that is effectively quasi-incompressible.

These findings indicate that gravitational scaling alone cannot generate sustained volumetric deformation within the stratified column under the conditions considered in this study. Instead, stratification acts as a stabilizing mechanism that suppresses long-term dilatational growth.

Further investigations could extend the model by incorporating rotational effects, dynamic density evolution, or stronger perturbations to examine whether additional mechanisms might produce more persistent deformation structures. Such extensions would improve the understanding of deformation processes in stratified geophysical fluid systems.

DATA AVAILABILITY

The numerical simulation data generated during this study are available from the corresponding author.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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