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Fixed point results of weakly polynomial contractions

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ABSTRACT

In this paper, a new family of weakly polynomial-type contractions defined on a metric space is presented. Under suitable hypotheses, it is shown that such contractive operators possess unique fixed points (FPs). Owing to the polynomial nature of the higher-order terms in the contractions, several significant particular cases, including existing results, are highlighted and discussed. In contrast to many existing Lipschitz-type inequalities, the proposed family of contractive inequalities does not force the mappings to be continuous.

Keywords: Fixed point, Weak contraction, Polynomial contraction.

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1. INTRODUCTION

Because of its role in real-world problems, nonlinear analysis has become an important area of mathematical research. Fixed point (FP) theory, a fundamental concept in this field, has been crucial in developing methods for solving various kinds of equations, including matrix, integral, and differential equations. See Refs. [1–3] for some of these applications.

The fundamental components of FP theory are the nonlinear operator whose FPs are to be determined, the metric that provides a structure on the ambient set, and the contractivity criterion that guarantees the existence of an FP. As shown in Refs. [4–14], these ideas have led to the development of several FP theorems during the last few decades. Since then, scholars have extended these fundamental tools to more generalized operators, such as multidimensional fixed points, and to increasingly abstract metric spaces (MSs). Deeper theoretical developments have resulted from improvements to contractivity conditions, which remain one of the main areas of study in this discipline. The foundation of metric FP theory is the Banach contraction mapping principle [15]. Alber and Guerre-Delabriere [16] presented an extension of this principle in Hilbert spaces. Similarly, Rhoades [17] developed the notion of weakly contractive mappings in metric spaces and provided a broader understanding of Ref. [16]. Two new classes of polynomial contractions were recently presented by Jleli *et al.* [18], who also proved FP theorems for these mappings. A novel class of mappings known as polynomial Kannan contractions was introduced by Moumen *et al.* [19], generalizing the idea in Ref. [18]. These mappings extend traditional Kannan contractions by incorporating higher-order polynomial terms. Shahi [20] also examined the central concept of Ref. [18] in the context of G-metric spaces.

According to the literature mentioned above, the hybrid concept of weakly polynomial contraction has not been thoroughly investigated. Inspired by earlier classes of contractive operators

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Table 1. Verification of Example 3.6.

(i,j)	$\varsigma_0(\Xi\aleph_i,\Xi\aleph_j)+\rho(\Xi\aleph_i,\Xi\aleph_j)$	$\varsigma_0(\aleph_i,\aleph_j)+\rho(\aleph_i,\aleph_j)-\phi(\varsigma_0(\aleph_i,\aleph_j)+\rho(\aleph_i,\aleph_j))$
$(1,2)$	4	4
$(1,3)$	2	2
$(1,4)$	0	1
$(2,3)$	4	4
$(2,4)$	4	5
$(3,4)$	2	2

[17, 18], this study introduces a new class of contraction mappings called weakly polynomial contractions. The main advantage of this novel family is that the polynomial terms allow several outcomes, including some well-known results in the literature, to be inferred. Furthermore, an FP result is established to illustrate the applicability and significance of these operators.

2. PRELIMINARIES

In this section, some fundamental notions used in the sequel are presented. Throughout, Ψ denotes the family of (c)-comparison functions; see Ref. [19].

Kannan [7] proposed one of the earliest generalizations of the Banach contraction as follows:

Definition 2.1 (Ref. [7])**.** Let (Θ,ρ) be an MS and $\Xi:\Theta\longrightarrow\Theta$ be a self-mapping. Ξ is said to be a Kannan contraction if there exists $\varrho\in\left[0,\frac{1}{2}\right)$ such that for all $\aleph,\hbar\in\Theta$,

$$\rho(\Xi\aleph,\Xi\hbar)\leq\varrho[\rho(\aleph,\Xi\aleph)+\rho(\hbar,\Xi\hbar)]. \tag{1}$$

Another well-known generalization of the Banach contraction principle was given in Ref. [21]. For such a contraction, existence and uniqueness of *FP* was investigated in the framework of a complete MS.

Definition 2.2 (Ref. [21])**.** Let (Θ,ρ) be an MS and $\Xi:\Theta\longrightarrow\Theta$ be a self-mapping. Then, Ξ is said to be a Reich contraction if there exist $a,b,c>0$ with $a+b+c<1$ such that for all $\aleph,\hbar\in\Theta$,

$$\rho(\Xi\aleph,\Xi\hbar)\leq a\rho(\aleph,\Xi\aleph)+b\rho(\hbar,\Xi\hbar)+c\rho(\aleph,\hbar). \tag{2}$$

With a slight alteration to the Kannan contraction, Chatterjea [22] obtained a new type of contraction as follows.

Definition 2.3 (Ref. [22])**.** Let (Θ,ρ) be an MS and $\Xi:\Theta\longrightarrow\Theta$ be self-mapping. Then Ξ is said to be a Chatterjea contraction if there exists $\varrho\in\left[0,\frac{1}{2}\right)$ such that for all $\aleph,\hbar\in\Theta$,

$$\rho(\Xi\aleph,\Xi\hbar)\leq\varrho[\rho(\aleph,\Xi\hbar)+\rho(\hbar,\Xi\aleph)]. \tag{3}$$

Definition 2.4 (Ref. [17])**.** A mapping $\Xi:\Theta\to\Theta$, where (Θ,ρ) is an MS, is said to be weakly contractive if

$$\rho(\Xi\aleph,\Xi\hbar)\leq\rho(\aleph,\hbar)-\varphi(\rho(\aleph,\hbar)), \tag{4}$$

where $\aleph,\hbar\in\Theta$ and $\varphi:[0,\infty)\to[0,\infty)$ is a continuous and nondecreasing function such that

$$\varphi(t)=0\quad\text{if and only if}\quad t=0.$$

Definition 2.5 (Ref. [18])**.** Let (Θ,ρ) be an MS and $\Xi:\Theta\to\Theta$ be given. Then Ξ is termed a polynomial contraction, if we can find $\varrho\in[0,1)$, $k\geq 1$, and

$$\beta_\iota:\Theta\times\Theta\to[0,\infty),\qquad \iota=0,1,\ldots,k.$$

$$\sum_{\iota=0}^{k}\beta_\iota(\Xi\aleph,\Xi\hbar)\rho^\iota(\Xi\aleph,\Xi\hbar)\ \leq\ \varrho\sum_{\iota=0}^{k}\beta_\iota(\aleph,\hbar)\rho^\iota(\aleph,\hbar), \tag{5}$$

for every $\aleph,\hbar\in\Theta$.

Definition 2.6 ([18])**.** Let (Θ,ρ) be an MS and $\Xi:\Theta\to\Theta$ be given. We say that Ξ is an *almost polynomial contraction* if we can find $\varrho\in[0,1)$, $k\geq 1$, $L=\{L_\iota\}_{\iota=0}^{k}\subset[0,\infty)$, and $\beta_\iota:\Theta\times\Theta\to[0,\infty)$, $\iota=0,\ldots,k$:

$$\sum_{\iota=0}^{k}\beta_\iota(\Xi\aleph,\Xi\hbar)\,\rho(\Xi\aleph,\Xi\hbar)\leq\varrho\sum_{\iota=0}^{k}\beta_\iota(\aleph,\hbar)\,\rho(\aleph,\hbar)+\sum_{\iota=0}^{k}L_\iota\,\rho(\hbar,\Xi\aleph). \tag{6}$$

for every $\aleph,\hbar\in\Theta$.

Definition 2.7 ([19])**.** A polynomial φ-contraction on an MS (Θ,ρ) is a map Ξ for which we can find $\varphi\in\Psi$, $k\in\mathbb{N}$, $\beta_\iota:\Theta\times\Theta\to[0,\infty)$:

$$\sum_{\iota=0}^{k}\beta_\iota(\Xi\aleph,\Xi\hbar)\rho^\iota(\Xi\aleph,\Xi\hbar)\ \leq\ \varphi\left(\sum_{\iota=0}^{k}\beta_\iota(\aleph,\hbar)\rho^\iota(\aleph,\hbar)\right),\quad \forall\,\aleph,\hbar\in\Theta.$$

3. MAIN RESULTS

We begin this section by introducing and examining a new class of weakly contractive mappings that involve higher-order polynomial terms. The presented main ideas are direct generalizations of the *FP* results due to Refs. [14, 17].

Definition 3.1. Let (Θ,ρ) be an MS and $\Xi:\Theta\to\Theta$ be given. Then Ξ is termed weakly polynomial contractive if we can find $k\geq 1$ and $\varsigma_\iota:\Theta\times\Theta\to[0,\infty)$, $\iota=0,\cdots,k$:

$$\sum_{\iota=0}^{k}\varsigma_\iota(\Xi\aleph,\Xi\hbar)\rho^\iota(\Xi\aleph,\Xi\hbar)\leq\sum_{\iota=0}^{k}\varsigma_\iota(\aleph,\hbar)\rho^\iota(\aleph,\hbar)-\phi\left(\sum_{\iota=0}^{k}\varsigma_\iota(\aleph,\hbar)\rho^\iota(\aleph,\hbar)\right), \tag{7}$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuous and monotonic nondecreasing function with $\phi(t)=0\Leftrightarrow t=0$.

Theorem 3.2. *Let (Θ,ρ) be a complete MS and $\Xi:\Theta\to\Theta$ be a weakly polynomial contractive mapping. Assume that $\ell\in\{1,\cdots,k\}$ and $E_\ell>0$ such that $\varsigma_\ell(\aleph,\hbar)\geq E_\ell$, $\aleph,\hbar\in\Theta$. Then, Ξ enjoys an FP in Θ.*

Proof. Let $\aleph_0\in\Theta$ be arbitrary but fixed. We define an iterative sequence $\{\aleph_n\}$ by

$$\aleph_1=\Xi\aleph_0,\ \aleph_2=\Xi\aleph_1=\Xi^2\aleph_0,\cdots,\ \aleph_n=\Xi\aleph_{n-1}=\Xi^n\aleph_0.$$

Letting $\aleph = \aleph_{n-1}$ and $\hbar = \aleph_n$, and using (7), we have

$$\sum_{\iota=0}^{k} \varsigma_\iota(\Xi\aleph_{n-1}, \Xi\aleph_n)\rho^\iota(\Xi\aleph_{n-1}, \Xi\aleph_n) \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)\rho^\iota(\aleph_{n-1}, \aleph_n) - \phi\left(\sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)\rho^\iota(\aleph_{n-1}, \aleph_n)\right).$$

This implies

$$\sum_{\iota=0}^{k} \varsigma_\iota(\aleph_n, \aleph_{n+1})\rho^\iota(\aleph_n, \aleph_{n+1}) \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)\rho^\iota(\aleph_{n-1}, \aleph_n) - \phi\left(\sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)\rho^\iota(\aleph_{n-1}, \aleph_n)\right). \tag{8}$$

Since for all $t \geq 0$, $\phi(t) \geq 0$, we have

$$\sum_{\iota=0}^{k} \varsigma_\iota(\aleph_n, \aleph_{n+1})\rho^\iota(\aleph_n, \aleph_{n+1}) \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)\rho^\iota(\aleph_{n-1}, \aleph_n).$$

Letting $S_n = \rho(\aleph_n, \aleph_{n+1})$, we have

$$\sum_{\iota=0}^{k} \varsigma_\iota(\aleph_n, \aleph_{n+1})S_n^\iota \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_{n-1}, \aleph_n)S_{n-1}^\iota. \tag{9}$$

Since

$$\varsigma_\iota(\aleph_n, \aleph_{n+1})\rho^\iota(\aleph_n, \aleph_{n+1}) \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_n, \aleph_{n+1})\rho^\iota(\aleph_n, \aleph_{n+1}),$$

that is, $\varsigma_\iota(\aleph_n, \aleph_{n+1})S_n^\iota \leq \sum_{\iota=0}^{k} \varsigma_\iota(\aleph_n, \aleph_{n+1})S_n^\iota$, then, from the hypothesis, $\varsigma_\ell(\aleph, \hbar) \geq E_\ell > 0$. And so, $\varsigma_\ell(\aleph_n, \aleph_{n+1})S_n^\ell \geq E_\ell S_n^\ell > 0$. Now (9) becomes $E_\ell S_n^\ell \leq E_\ell S_{n-1}^\ell$, where $S_{n-1} = \rho(\aleph_{n-1}, \aleph_n)$. This implies:

$$\rho(\aleph_n, \aleph_{n+1}) \leq \rho(\aleph_{n-1}, \aleph_n). \tag{10}$$

It follows that the sequence $\{\rho(\aleph_n, \aleph_{n+1})\}$ is monotone decreasing and converges to a point say p. That is,

$$\rho(\aleph_n, \aleph_{n+1}) \to p \quad \text{as} \quad n \to \infty. \tag{11}$$

We now show that $p = 0$. Now, letting $n \to \infty$ in (8), we have $E_\ell p \leq E_\ell p - \phi(E_\ell p)$ which implies that $\phi(E_\ell p) \leq 0$ if and only if $E_\ell p = 0$. But $E_\ell > 0$ and so it follows that $p = 0$. Hence,

$$\rho(\aleph_n, \aleph_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

To demonstrate that $\{\aleph_n\}$ is fundamental. Let $\{\aleph_n\}$ not be true, then we can find $\varepsilon > 0$ with $\{\aleph_{m(k)}\}$ and $\{\aleph_{n(k)}\}$ fulfilling $n(k) > m(k) > k$:

$$\rho(\aleph_{m(k)}, \aleph_{n(k)}) \geq \varepsilon. \tag{13}$$

Now, we pick $n(k)$ in such a way that it is the smallest of all $n(k) > m(k) > k$ fulfilling (13):

$$\rho(\aleph_{m(k)}, \aleph_{n(k)-1}) < \varepsilon. \tag{14}$$

Then it follows that

$$\varepsilon \leq \rho(\aleph_{m(k)}, \aleph_{n(k)}) \leq \rho(\aleph_{m(k)}, \aleph_{n(k)-1}) + \rho(\aleph_{n(k)-1}, \aleph_{n(k)}) < \varepsilon + \rho(\aleph_{n(k)-1}, \aleph_{n(k)}). \tag{15}$$

Letting $k \to \infty$ and using (12), we have

$$\lim_{k\to\infty} \rho(\aleph_{m(k)}, \aleph_{n(k)}) = \varepsilon + 0 = \varepsilon. \tag{16}$$

Again,

$$\rho(\aleph_{n(k)-1}, \aleph_{m(k-1)}) \leq \rho(\aleph_{m(k)-1}, \aleph_{n(k)}) + \rho(\aleph_{n(k)}, \aleph_{m(k)}) + \rho(\aleph_{m(k)}, \aleph_{m(k)-1}). \tag{17}$$

Letting $k \to \infty$ and using (12), (16), we obtain

$$\lim_{k\to\infty} \rho(\aleph_{n(k)-1}, \aleph_{m(k)-1}) = 0 + \varepsilon + 0 = \varepsilon. \tag{18}$$

Now, letting $\aleph = \aleph_{m(k)-1}$ and $\hbar = \aleph_{n(k-1)}$, and using (13), we have

$$E_\ell\rho^\ell(\aleph_{m(k)}, \aleph_{n(k)}) \leq E_\ell\rho^\ell(\aleph_{m(k)-1}, \aleph_{n(k)-1}) - \phi(E_\ell\rho^\ell(\aleph_{m(k)-1}, \aleph_{n(k)-1})). \tag{19}$$

Letting $k \to \infty$ and using (16) and (18), we have $E_\ell\varepsilon \leq E_\ell\varepsilon - \phi(E_\ell\varepsilon)$, which implies $\phi(E_\ell\varepsilon) \leq 0$ if and only if $E_\ell\varepsilon = 0$. This implies that $\varepsilon = 0$. But this is a contradiction, since $\varepsilon > 0$ and therefore $\{\aleph_n\}$ is a Cauchy sequence. Since (Θ, ρ) is a complete MS, the sequence $\{\aleph_n\}$ converges to a point say μ in Θ. That is,

$$\lim_{n\to\infty} \aleph_n = \mu. \tag{20}$$

We now show that μ is an *FP* of Ξ. Now, letting $\aleph = \aleph_n$ and $\hbar = \mu$, we have

$$E_\ell\rho^\ell(\Xi\aleph_n, \Xi\mu) \leq E_\ell\rho^\ell(\aleph_n, \mu) - \phi(E_\ell\rho^\ell(\aleph_n, \mu)). \tag{21}$$

Letting $n \to \infty$ in (21), we have $E_\ell\rho^\ell(\mu, \Xi\mu) \leq E_\ell\rho^\ell(\mu, \mu) - \phi(E_\ell\rho^\ell(\mu, \mu))$, which implies $E_\ell\rho^\ell(\mu, \Xi\mu) \leq 0 - \phi(0) = 0$, and so $\rho^\ell(\mu, \Xi\mu) = 0$, yielding $\rho(\mu, \Xi\mu) = 0$. Therefore, $\mu = \Xi\mu$, implying that μ is a *FP* of Ξ. We now show that μ is a unique *FP* of Ξ. Suppose that u and v are two *FPs* of Ξ, that is, $\mu = \Xi\mu$ and $v = \Xi v$ with $\mu \neq v$. Then,

$$\sum_{i=0}^{k} \varsigma_\iota(\Xi u, \Xi v)\rho^\iota(\Xi u, \Xi v) \leq \sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v) - \phi\left(\sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v)\right),$$

and so $\sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v) \leq \sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v) - \phi\left(\sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v)\right)$, that is, $\phi\left(\sum_{i=0}^{k} \varsigma_\iota(u, v)\rho^\iota(u, v)\right) \leq 0$, which implies $E_\ell\rho^\ell(u, v) = 0$, leading to $\rho(u, v) = 0$, and so $u = v$. Hence, Ξ has a unique *FP*. □

Corollary 3.3 ([17]). *Let (Θ, ρ) be a complete MS. If $\Xi : \Theta \to \Theta$ is a weakly contractive mapping, then Ξ has a unique FP.*

Proof. Take $k = 1, \varsigma_1 = 1$ and $\varsigma_0 = 0$ in Theorem 3.2. □

Corollary 3.4. *Let (Θ,ρ) be a complete MS and $\varsigma_0 : \Theta\times\Theta \to [0,\infty)$, $\Xi : \Theta \to \Theta$ be any two mappings satisfying:*

$$\varsigma_0(\Xi\aleph,\Xi\hbar) + \rho(\Xi\aleph,\Xi\hbar) \le \varsigma_0(\aleph,\hbar) + \rho(\aleph,\hbar),\ \ \aleph,\hbar \in \Theta.$$

Assume further that there exists $A_0 > 0$ such that $\varsigma_0(\aleph,\hbar) \ge A_0$, then Ξ has a unique FP in Θ.

Proof. Take $k = 1, \varsigma_1 = 1$ and $\phi(t) = 0$ for all $t \ge 0$ in Theorem 3.2. □

Corollary 3.5 ([18]). *Let (Θ,ρ) be a complete MS and $\varsigma_\iota : \Theta\times\Theta \to [0,\infty)$, $\Xi : \Theta \to \Theta$ be any two mappings. Assume that there exist $\ell \in \{1,\cdots,k\}$, $E_\ell > 0$ and $\varrho \in [0,1)$ such that $\varsigma_\ell(\aleph,\hbar) \ge E_\ell$ and $\forall\ \aleph,\hbar \in \Theta$,*

$$\sum_{\iota=0}^{k} \varsigma_\iota(\Xi\aleph,\Xi\hbar)\rho^\iota(\Xi\aleph,\Xi\hbar) \le \varrho \sum_{\iota=0}^{k} \varsigma_\iota(\aleph,\hbar)\rho^\iota(\aleph,\hbar).$$

Then Ξ has a unique FP in Θ.

Proof. Take $\phi(t) = (1-\varrho)t,\ t \ge 0$ in Theorem 3.2. □

In the following, we construct a comparative example to support the main assumptions of Theorem 3.2 and indicate its distinctness from the principal idea of Ref. [17].

Example 3.6. Let $\Theta = \{\aleph_1,\aleph_2,\aleph_3,\aleph_4\}$ and $\Xi : \Theta \to \Theta$ be the mapping defined by

$$\Xi\aleph_1 = \aleph_1,\quad \Xi\aleph_2 = \aleph_3,\quad \Xi\aleph_3 = \aleph_4,\quad \Xi\aleph_4 = \aleph_1.$$

Let ρ be the discrete metric on Θ, i.e.

$$\rho(\aleph_i,\aleph_j) = \begin{cases} 1 & \text{if } i \ne j, \\ 0 & \text{if } i = j. \end{cases},$$

Take $\phi(t) = \frac{t}{2},\quad t > 0$. Consider the mapping $\varsigma_0 : \Theta\times\Theta \to [0,\infty)$ defined by

$$\begin{aligned} \varsigma_0(\aleph_i,\aleph_j) &= \varsigma_0(\aleph_j,\aleph_i),\\ \varsigma_0(\aleph_i,\aleph_i) &= 0,\\ \varsigma_0(\aleph_1,\aleph_2) &= \varsigma_0(\aleph_2,\aleph_3) = 7,\\ \varsigma_0(\aleph_1,\aleph_3) &= \varsigma_0(\aleph_3,\aleph_4) = 3,\\ \varsigma_0(\aleph_1,\aleph_4) &= 1,\\ \varsigma_0(\aleph_2,\aleph_4) &= 9. \end{aligned}$$

We claim that

$$\begin{aligned} \varsigma_0(\Xi\aleph,\Xi\omega) + \rho(\Xi\aleph,\Xi\omega) \le\ & \varsigma_0(\aleph,\omega) + \rho(\aleph,\omega)\\ & - \phi(\varsigma_0(\aleph,\omega) + \rho(\aleph,\omega)) \end{aligned}$$

for every $\aleph,\omega \in \Theta$. That is, Ξ is a weakly polynomial contraction in the sense of Definition 3.1 with $k = 1$ and $\varsigma_1 = 1$. We now show that the claim holds for all $\aleph_i,\aleph_j \in \Theta$ with $i \ne j$. Now, for $(i,j) = (1,2)$, we have

$$\begin{aligned} \varsigma_0(\Xi\aleph_1,\Xi\aleph_2) + \rho(\Xi\aleph_1,\Xi\aleph_2) \le\ & \varsigma_0(\aleph_1,\aleph_2) + \rho(\aleph_1,\aleph_2)\\ & - \phi(\varsigma_0(\aleph_1,\aleph_2) + \rho(\aleph_1,\aleph_2)), \end{aligned}$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_1,\aleph_3) + \rho(\aleph_1,\aleph_3) \le\ & \varsigma_0(\aleph_1,\aleph_2) + \rho(\aleph_1,\aleph_2)\\ & - \phi(\varsigma_0(\aleph_1,\aleph_2) + \rho(\aleph_1,\aleph_2)),\\ 3 + 1 \le\ & 7 + 1 - \phi(7+1),\\ \le\ & 8 - \frac{8}{2}, \end{aligned}$$

and so $4 \le 4$. For $(i,j) = (1,3)$, we have

$$\begin{aligned} \varsigma_0(\Xi\aleph_1,\Xi\aleph_3) + \rho(\Xi\aleph_1,\Xi\aleph_3) \le\ & \varsigma_0(\aleph_1,\aleph_3) + \rho(\aleph_1,\aleph_3)\\ & - \phi(\varsigma_0(\aleph_1,\aleph_3) + \rho(\aleph_1,\aleph_3)), \end{aligned}$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_1,\aleph_4) + \rho(\aleph_1,\aleph_4) \le\ & \varsigma_0(\aleph_1,\aleph_3) + \rho(\aleph_1,\aleph_3)\\ & - \phi(\varsigma_0(\aleph_1,\aleph_3) + \rho(\aleph_1,\aleph_3)),\\ 1 + 1 \le\ & 3 + 1 - \phi(3+1),\\ \le\ & 4 - \frac{4}{2}, \end{aligned}$$

and so $2 \le 2$. For $(i,j) = (2,3)$, we have

$$\begin{aligned} \varsigma_0(\Xi\aleph_2,\Xi\aleph_3) + \rho(\Xi\aleph_2,\Xi\aleph_3) \le\ & \varsigma_0(\aleph_2,\aleph_3) + \rho(\aleph_2,\aleph_3)\\ & - \phi(\varsigma_0(\aleph_2,\aleph_3) + \rho(\aleph_2,\aleph_3)), \end{aligned}$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_3,\aleph_4) + \rho(\aleph_3,\aleph_4) \le\ & \varsigma_0(\aleph_2,\aleph_3) + \rho(\aleph_2,\aleph_3)\\ & - \phi(\varsigma_0(\aleph_2,\aleph_3) + \rho(\aleph_2,\aleph_3)),\\ 3 + 1 \le\ & 7 + 1 - \phi(7+1),\\ \le\ & 8 - \frac{8}{2}, \end{aligned}$$

and so $4 \le 4$. For $(i,j) = (2,4)$, we have

$$\begin{aligned} \varsigma_0(\Xi\aleph_2,\Xi\aleph_4) + \rho(\Xi\aleph_2,\Xi\aleph_4) \le\ & \varsigma_0(\aleph_2,\aleph_4) + \rho(\aleph_2,\aleph_4)\\ & - \phi(\varsigma_0(\aleph_2,\aleph_4) + \rho(\aleph_2,\aleph_4)), \end{aligned}$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_3,\aleph_1) + \rho(\aleph_3,\aleph_1) \le\ & \varsigma_0(\aleph_2,\aleph_4) + \rho(\aleph_2,\aleph_4)\\ & - \phi(\varsigma_0(\aleph_2,\aleph_4) + \rho(\aleph_2,\aleph_4)),\\ 3 + 1 \le\ & 9 + 1 - \phi(9+1),\\ \le\ & 10 - \frac{10}{2}, \end{aligned}$$

and so $4 \le 5$. For $(i,j) = (3,4)$, we have

$$\begin{aligned} \varsigma_0(\Xi\aleph_3,\Xi\aleph_4) + \rho(\Xi\aleph_3,\Xi\aleph_4) \le\ & \varsigma_0(\aleph_3,\aleph_4) + \rho(\aleph_3,\aleph_4)\\ & - \phi(\varsigma_0(\aleph_3,\aleph_4) + \rho(\aleph_3,\aleph_4)), \end{aligned}$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_4,\aleph_1) + \rho(\aleph_4,\aleph_1) \le\ & \varsigma_0(\aleph_3,\aleph_4) + \rho(\aleph_3,\aleph_4)\\ & - \phi(\varsigma_0(\aleph_3,\aleph_4) + \rho(\aleph_3,\aleph_4)),\\ 1 + 1 \le\ & 3 + 1 - \phi(3+1),\\ \le\ & 4 - \frac{4}{2}, \end{aligned}$$

and so $2 \leq 2$. For $(i, j) = (1, 4)$, we have

$$\varsigma_0(\Xi\aleph_1, \Xi\aleph_4) + \rho(\Xi\aleph_1, \Xi\aleph_4) \leq \varsigma_0(\aleph_1, \aleph_4) + \rho(\aleph_1, \aleph_4) - \phi(\varsigma_0(\aleph_1, \aleph_4) + \rho(\aleph_1, \aleph_4)),$$

which implies

$$\begin{aligned} \varsigma_0(\aleph_1, \aleph_1) + \rho(\aleph_1, \aleph_1) &\leq \varsigma_0(\aleph_1, \aleph_4) + \rho(\aleph_1, \aleph_4) - \phi(\varsigma_0(\aleph_1, \aleph_4) + \rho(\aleph_1, \aleph_4)), \\ 0 + 0 &\leq 1 + 1 - \phi(1 + 1), \\ &\leq 2 - \frac{2}{2}, \end{aligned}$$

and so $0 \leq 1$. The verification is summarized in Table 1.

Hence, all the assumptions of Theorem 3.2 are satisfied. Therefore, Ξ has a unique FP given by $\aleph_1 = \Xi\aleph_1$.

We now see that the FP theorem of weakly contractive mapping due to [17] is not applicable in this example. To see this, let $\aleph^* = \aleph_1$ and $\omega^* = \aleph_2$. Then,

$$\rho(\aleph_1, \aleph_2) = 1, \quad \rho(\Xi\aleph_1, \Xi\aleph_2) = \rho(\aleph_2, \aleph_3) = 1$$

$$\phi(\rho(\aleph_1, \aleph_2)) = \frac{\rho(\aleph_1, \aleph_2)}{2} = \frac{1}{2}, \quad \text{Therefore:}$$

$$\rho(\Xi\aleph_1, \Xi\aleph_2) = 1, \quad \rho(\aleph_1, \aleph_2) - \phi(\rho(\aleph_1, \aleph_2)) = 1 - \frac{1}{2} = \frac{1}{2}$$

That is, $\rho(\Xi\aleph_1, \Xi\aleph_2) > \rho(\aleph_1, \aleph_2) - \phi(\rho(\aleph_1, \aleph_2))$. This shows that Ξ is not a weakly contractive mapping in the sense of Rhoades [17].

4. CONCLUSION

The hybrid *FP* theorem presented in this paper, along with the obtained corollaries, demonstrated that many existing *FP* concepts can be derived as special cases of our main result. This unifying approach is significant because it reduces the need to prove multiple independent theorems, instead, showing how various contraction conditions are interconnected within a broader theoretical framework.

The distance metrics and inequalities established in our analysis also provide estimates on the rate of convergence of iterative sequences to the fixed point, which is valuable for numerical implementations. This bridges the gap between pure theoretical results and computational applications.

Several directions for future work emerged from this paper. These include extending the results to generalized MS, investigating weakly polynomial contractions for multivalued mappings, and exploring applications to fractional differential equations and nonlinear systems. The framework developed here provides a solid foundation for such extensions.

DATA AVAILABILITY

We do not have any research data outside the submitted manuscript file.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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